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GLOBAL WEAK SOLUTIONS TO THE STOCHASTIC ERICKSEN–LESLIE SYSTEM IN DIMENSION TWO

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Abstract. We establish the global existence of weak martingale solutions to the simplified stochastic Ericksen–Leslie system modeling the nematic liquid crystal flow driven by Wiener-type noises on the two-dimensional bounded domains. The construction of solutions is based on the convergence of Ginzburg–Landau approximations. To achieve such a convergence, we first utilize the concentration-cancellation method for the Ericksen stress tensor fields based on a Pohozaev type argument, and then the Skorokhod compactness theorem, which is built upon uniform energy estimates.

1. **Introduction.** In this article, we consider the following simplified stochastic Ericksen–Leslie system on a two dimensional bounded domain D with smooth boundary:

$$\begin{cases} d\mathbf{u} + (\mathbf{u}\cdot\nabla\mathbf{u} + \nabla P - \mu\Delta\mathbf{u})dt = -\lambda\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d})dt + \xi_1 S(\mathbf{u})dW_1, \\ \nabla\cdot\mathbf{u} = 0, \\ d\mathbf{d} + \mathbf{u}\cdot\nabla\mathbf{d}dt = \gamma(\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d})dt + \xi_2(\mathbf{d}\times\mathbf{h})\circ dW_2, \end{cases} \tag{1}$$

where $\mathbf{u} : D\times\mathbb{R}_+\times\Omega\to\mathbb{R}^2$, $\mathbf{d} : D\times\mathbb{R}_+\times\Omega\to\mathbb{S}^2$ represent the fluid velocity field and the molecular director field, respectively, $P : D\times\mathbb{R}_+\times\Omega\to\mathbb{R}$ stands for the hydro-static pressure. $(\nabla\mathbf{d}\odot\nabla\mathbf{d})_{ij} = \langle\partial_i\mathbf{d}, \partial_j\mathbf{d}\rangle$ $(1\le i,j\le 2)$ represents the Ericksen stress tensor field. The multiplicative noise term $S(\mathbf{u})dW_1$ in $(1)_1$ shall be understood in the Itô sense with a cylindrical Wiener process W_1 on a separable Hilbert space K_1. For a given $\mathbf{h} : \mathbb{R}^2\to\mathbb{R}^3$, $(\mathbf{d}\times\mathbf{h})\circ dW_2$ is understood in the Stratonovich sense with a standard real-valued Brownian motion W_2. $\mu, \lambda, \gamma, \xi_1, \xi_2$ are positive physical constants.

We assume, further, $(\mathbf{u}, \mathbf{d})$ satisfies the following initial-boundary conditions:

$$(\mathbf{u}, \mathbf{d})|_{t=0} = (\mathbf{u}_0, \mathbf{d}_0), \quad \text{in } D. \tag{2}$$

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$$\mathbf{u}|_{\partial D}=0, \quad \left.\frac{\partial \mathbf{d}}{\partial \mathbf{n}}\right|_{\partial D}=0, \quad (\text{or } \mathbf{d}|_{\partial D}=\mathbf{d}_0). \tag{3}$$

where $\mathbf{n}$ is the unit outward normal to ∂D. In this paper, we use the Ginzburg–Landau type approximation which relaxes the condition $|\mathbf{d}|=1$ in (1) by introducing a penalized term. More specifically, we have a family of solutions $(\mathbf{u}^\varepsilon,\mathbf{d}^\varepsilon)_{0<\varepsilon<1}$ to

$$\begin{cases} d\mathbf{u}^\varepsilon+(\mathbf{u}^\varepsilon\cdot\nabla\mathbf{u}^\varepsilon+\nabla P^\varepsilon-\mu\Delta\mathbf{u}^\varepsilon)dt \\ =-\lambda\nabla\cdot(\nabla\mathbf{d}^\varepsilon\odot\nabla\mathbf{d}^\varepsilon)dt+\xi_1 S(\mathbf{u}^\varepsilon)dW_1, \\ \nabla\cdot\mathbf{u}^\varepsilon=0, \\ d\mathbf{d}^\varepsilon+\mathbf{u}^\varepsilon\cdot\nabla\mathbf{d}^\varepsilon dt=\gamma\Big(\Delta\mathbf{d}^\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)\Big)dt+\xi_2(\mathbf{d}^\varepsilon\times\mathbf{h})\circ dW_2, \end{cases} \tag{4}$$

where $\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)=\nabla_{\mathbf{d}}F_\varepsilon(\mathbf{d}^\varepsilon)=\frac{1}{\varepsilon^2}(|\mathbf{d}^\varepsilon|^2-1)\mathbf{d}^\varepsilon$ with $F_\varepsilon(\mathbf{d})=\frac{1}{4\varepsilon^2}(1-|\mathbf{d}|^2)^2$.

In the deterministic case ($\xi_1=\xi_2=0$), the global existence of the weak solutions to the Ginzburg–Landau type Ericksen–Leslie system (4), which is a simplified version of the full Ericksen–Leslie system [11, 12, 19, 20], was first investigated by Lin–Liu [22]. For the simplified Ericksen–Leslie system (1), motivated by Struwe [27] on harmonic map heat flows in dimension two, the existence of a unique global weak solution with partial regularity was established Lin–Lin–Wang [21] and Lin-Wang [23], which was generalized by Huang–Lin–Wang [17] for the full Ericksen–Leslie system. See also Hong [15] and Hong–Xin [16] for related works. We refer the readers to [24] for a comprehensive survey for the recent developments. The question that whether one can obtain a weak solution of (1) via sending $\varepsilon\to 0$ in (4) remains open due to the difficulty with possible defect measures appearing in the Ericksen stress tensor field. In a very recent paper [18], Kortum applied a concentration-cancellation method initiated by Diperna–Majda [9] on the 2-D incompressible Euler equation to show that $\operatorname{div}(\nabla\mathbf{d}^\varepsilon\odot\nabla\mathbf{d}^\varepsilon)\rightharpoonup\operatorname{div}(\nabla\mathbf{d}\odot\nabla\mathbf{d})$ in the torus $\mathbb{T}^2$. For general domains and full Ericksen–Leslie system, the weak compactness result was shown in [10] via the Hopf differential and the Pohozaev technique. We also want to point out that in 3-D, the Ginzburg–Landau approximation was implemented in [25] to construct a global weak solutions to the simplified Ericksen–Leslie system (1) with the half-sphere assumption imposed on directors ($\mathbf{d}\in\mathbb{S}^2_+$).

On the other hand, there is a growing number of research studies that are devoted to the simplified stochastic Ericksen–Leslie system (4) with various types of random noises ($\xi_1^2+\xi_2^2>0$). See for instance, [3, 5, 6, 7]. For the mathematical modeling, taking the stochastic terms into account reflects the influence of environmental noises, the measurement uncertainties as well as the thermal fluctuations. Analogously, Bouard–Hocquet–Prohl obtained the Struwe-like global solution to (1) in [8] by a bootstrap argument together with Gyöngy–Krylov L^p estimates [14]. Very recently, Brzeźniak, Deugoué, and Razafimandimby in [2] proved the existence of short time strong solutions to the simplified stochastic Ericksen–Leslie system. The main goal of this paper is to obtain a global weak solution to (1) by extending the compactness argument from [10] into the stochastic setting.

For simplicity, we assume $\lambda=\xi_1=\gamma=\xi_2=1$. We introduce some function spaces:

$$\mathbf{H}=\text{closure of } C_0^\infty(D,\mathbb{R}^2)\cap\{f|\nabla\cdot f=0\} \text{ in } L^2(D,\mathbb{R}^2),$$
$$\mathbf{J}=\text{closure of } C_0^\infty(D,\mathbb{R}^2)\cap\{f|\nabla\cdot f=0\} \text{ in } H_0^1(D,\mathbb{R}^2),$$
$$H^1(D,\mathbb{S}^2)=\left\{f\in H^1(D,\mathbb{R}^3)||f|=1 \text{ a.e. } x\in D\right\}.$$

For a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, let K_1 be an infinite dimensional separable Hilbert space and $W_1 = \{W_1(t)\}_{t\geq 0}$ be a K_1-cylindrical Wiener process such that it is formally written as a series

$$W_1(t) = \sum_{i=1}^{\infty} B_i(t) e_i, \forall t \geq 0,$$

where $\{B_i(t)\}_{i=1}^{\infty}$ is a family of i.i.d. standard Brownian motions and $\{e_i\}_{i=1}^{\infty}$ is an orthonormal base of K_1. The above series does not converge in K_1, but it does converge in K_2 if K_2 is a larger Hilbert space containing K_1 such that the inclusion map $J : K_1 \to K_2$ is Hilbert-Schmidt. It is always possible to construct a space K_2 with this property. For example, we can define K_2 to be the closure of K_1 under the norm

$$\|x\|_{K_2}^2 = \sum_{i=1}^{\infty} \frac{1}{i^2} \langle x, e_i \rangle_{K_1}^2.$$

Then we can view W_1 as a K_2-valued Wiener process. Let $W_2 = \{W_2(t)\}_{t\geq 0}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ adapted to $\{\mathcal{F}_t\}_{t\geq 0}$. S is a map from $\mathbf{H}$ to $\mathcal{L}_2(K_1, \mathbf{J})$, where $\mathcal{L}_2(K_1, \mathbf{J})$ denotes the space of all Hilbert–Schmidt operators from K_1 to $\mathbf{J}$, i.e., $\sum_{i=1}^{\infty} \|S(\cdot)(e_i)\|_{\mathbf{J}}^2 < \infty$, if $\{e_i\}_{i=1}^{\infty}$ is an orthonormal base of K_1.

We now introduce the notion of a weak martingale solution to (1).

Definition 1.1. A weak martingale solution to (1), (2), (3) is a system consisting of a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, and $\mathcal{F}_t$ adapted stochastic processes $(\mathbf{u}(t), \mathbf{d}(t), W_1(t), W_2(t))_{t\geq 0}$ such that for any $0 < T < \infty$

1. $\{W_1(t)\}_{t\geq 0}$ (or $\{W_2(t)\}_{t\geq 0}$) is a K_1-cylindrical (resp. real-valued) Wiener process.
2. $(\mathbf{u}, \mathbf{d}) : \Omega \times \mathbb{R}_+ \to \mathbf{H} \times H^1(D, \mathbb{S}^2)$ is progressively measurable with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that for almost surely $\omega \in \Omega$,
$$\mathbf{u} \in L_t^\infty([0, T], \mathbf{H}) \cap L_t^2([0, T], \mathbf{J}), \quad \mathbf{d} \in L^2([0, T], H^1(D, \mathbb{S}^2)).$$
3. We have
$$\mathbb{E}\left[\sup_{0\leq t\leq T} \int_{D\times\{t\}} |\mathbf{u}|^2 + |\nabla \mathbf{d}|^2 + \int_0^T \int_D (|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}|^2) dx ds\right] < \infty. \tag{5}$$
Here $\mathbb{E}$ stands for the expectation.
4. For almost surely $\omega \in \Omega$, for every $t \in [0, T]$, for any $\varphi \in C^\infty(D, \mathbb{R}^2)$, $\operatorname{div} \varphi = 0$, we have
$$\begin{aligned} &- \int_{D\times\{t\}} \langle \mathbf{u}, \varphi \rangle dx - \int_0^t \int_D (\langle \mathbf{u} \otimes \mathbf{u}, \nabla \varphi \rangle + \langle \mathbf{u}, \Delta \varphi \rangle) dx ds \\ &= - \int_D \langle \mathbf{u}_0, \varphi \rangle \, dx + \int_0^t \int_D (\langle \nabla \mathbf{d} \odot \nabla \mathbf{d} - \frac{1}{2} |\nabla \mathbf{d}|^2 \mathbb{I}_2, \nabla \varphi \rangle) dx ds \\ &+ \int_0^t \int_D \langle \varphi, S(\mathbf{u}) dW_1(s) \rangle dx, \end{aligned} \tag{6}$$
and for any $\psi \in C^\infty(D, \mathbb{R}^3)$,

$$
\begin{aligned}
&-\int_{D\times\{t\}}\langle \mathbf{d},\psi\rangle dx-\int_0^t\int_D(\langle \mathbf{u}\otimes\mathbf{d},\nabla\psi\rangle+\langle\mathbf{d},\Delta\psi\rangle)dxds\\
&=-\int_D\langle\mathbf{d}_0,\psi\rangle dx+\int_0^t\int_D\langle|\nabla\mathbf{d}|^2\mathbf{d},\psi\rangle dxds \qquad (7)\\
&+\int_0^t\int_D\langle\psi,(\mathbf{d}\times\mathbf{h})\rangle dx\circ dW_2(s).
\end{aligned}
$$

We introduce the following assumptions that are required by our main theorem.

Assumption 1. *Let $S:\mathbf{H}\to\mathcal{L}_2(K_1,\mathbf{J})$ be a global Lipschitz map. In particular, there exists $C>0$ such that $\|S(\mathbf{u})\|^2_{\mathcal{L}_2(K_1,\mathbf{J})}\le C(1+\|\mathbf{u}\|^2_{\mathbf{H}})$ for all $\mathbf{u}\in\mathbf{H}$. $\mathbf{h}\in H^2(\mathbb{R}^2,\mathbb{R}^3)$ and $(\mathbf{u}_0,\mathbf{d}_0)\in\mathbf{H}\times H^1(D;\mathbb{S}^2)$. Furthermore, we assume $\{(\mathbf{u}_0^\varepsilon,\mathbf{d}_0^\varepsilon)\}_{0<\varepsilon<1}\subset\mathbf{J}\times H^2(D;\mathbb{S}^2)$ and satisfies $(\mathbf{u}_0^\varepsilon,\mathbf{d}_0^\varepsilon)\to(\mathbf{u}_0,\mathbf{d}_0)$ in $\mathbf{H}\times H^1(D;\mathbb{R}^3)$.*

Similar to Definition 1.1, a weak martingale solution $(\mathbf{u}^\varepsilon(t),\mathbf{d}^\varepsilon(t),W_1^\varepsilon(t),W_2^\varepsilon(t))$ adapted to a family of complete filtered probability spaces $(\Omega^\varepsilon,\mathcal{F}^\varepsilon,\mathbb{P}^\varepsilon,\{\mathcal{F}_t^\varepsilon\}_{t\ge0})$ to (4), (2), (3) can be defined. Under Assumption 1, the existence of weak martingale solutions $(\mathbf{u}^\varepsilon,\mathbf{d}^\varepsilon,W_1^\varepsilon,W_2^\varepsilon)$ with respect to $(\Omega^\varepsilon,\mathcal{F}^\varepsilon,\mathbb{P}^\varepsilon,\{\mathcal{F}_t^\varepsilon\}_{t\ge0})$ was established in [6, Theorem 3.2] via the Faedo–Galerkin approximation and the weak compactness method, together with the path-wise uniqueness in 2-D [6, Theorem 3.4]. It has been proved in the recent work [4, Theorem 3.17] that (4) possesses a unique strong solution, that is, given $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\ge0},W_1,W_2)$, there exists a unique pair of stochastic processes $(\mathbf{u}^\varepsilon,\mathbf{d}^\varepsilon)$ which solves (4) with respect to $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\ge0},W_1,W_2)$ for initial data $(\mathbf{u}_0^\varepsilon,\mathbf{d}_0^\varepsilon)\in\mathbf{J}\times H^2(D;\mathbb{R}^3)$.

Our main result asserts the existence of a global weak martingale solution to (1) via passing the limit of solutions $(\mathbf{u}^\varepsilon,\mathbf{d}^\varepsilon)$ to (4):

Theorem 1.2. *Under Assumption 1, there exist a completed filtered probability space $(\Omega',\mathcal{F}',\mathbb{P}')$ and a sequence of weak martingale solutions $(\overline{\mathbf{u}}^\varepsilon,\overline{\mathbf{d}}^\varepsilon,\overline{W}_1^\varepsilon,\overline{W}_2^\varepsilon)$ to (4), (2), (3) on $(\Omega',\mathcal{F}',\mathbb{P}')$ and a weak martingale solution $(\mathbf{u},\mathbf{d},W_1',W_2')$ to (1), (2), (3) such that after passing to a subsequence,*

$$
\overline{\mathbf{u}}^\varepsilon\rightharpoonup\mathbf{u}\ \text{in}\ L^2(\Omega';L^2([0,T],H^1(D))),\quad \overline{\mathbf{d}}^\varepsilon\rightharpoonup\mathbf{d}\ \text{in}\ L^2(\Omega';L^2([0,T],H^1(D)))
$$

as $\varepsilon\to0$.

We would like to remark that it is feasible to extend the conclusion of Theorem 1.2 to dimension three, if, in addition, the initial data $d_0(\Omega)\subset\mathbb{S}^2_+$ and h satisfies $h\times e_3=0$, where $e_3=(0,0,1)^T$. In this case, one can first show that the third component of d^ε, $(d^\varepsilon)^3\ge0$ in $\Omega\times\mathbb{R}_+$, and then modify the argument by Lin-Wang [25] from the deterministic to stochastic case accordingly.

The paper is organized as follows. In section 2 we establish some uniform energy estimates for the approximate solutions $(\mathbf{u}^\varepsilon,\mathbf{d}^\varepsilon)$ by Itô's formula. The convergence of the approximate system, in particular, the Ericksen stress tensor field and martingale terms will be discussed in section 3. In Appendix A, we provide the computation of Itô's formula for two functionals of $\mathbf{d}$.

2. **Uniform estimates on approximated solutions.**

In this section, we will derive an uniform energy estimate for (4), (2), (3) via Itô's calculus.

For simplicity, we denote $\|\cdot\|:=\|\cdot\|_{L^2(D)}$. First, applying Itô's formula to $\frac12\|\mathbf{u}^\varepsilon(t)\|^2$ yields

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{u}^\varepsilon(t)\|^2 - \frac{1}{2}\|\mathbf{u}_0^\varepsilon\|^2 + \int_0^t \int_D |\nabla \mathbf{u}^\varepsilon|^2 dxds \qquad (8)\\
&= \int_0^t \int_D \langle \nabla \mathbf{d}^\varepsilon \odot \nabla \mathbf{d}^\varepsilon, \nabla \mathbf{u}^\varepsilon \rangle dxds\\
&+ \frac{1}{2}\int_0^t \|S(\mathbf{u}^\varepsilon)\|^2_{\mathcal{L}_2(K_1,\mathbf{H})}\, ds + \int_0^t \int_D \langle \mathbf{u}^\varepsilon, S(\mathbf{u}^\varepsilon) dW_1(s)\rangle dx,
\end{aligned}$$

where we have used the cancellation property

$$\int_0^t \int_D \langle \mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon \rangle dxds = 0.$$

From the relation between Stratonovich and Itô's integral, we have that

$$(\mathbf{d}\times\mathbf{h})\circ dW_2 = \frac{1}{2}((\mathbf{d}\times\mathbf{h})\times\mathbf{h})dt + (\mathbf{d}\times\mathbf{h})dW_2.$$

Therefore $(4)_3$ can be written as

$$d\mathbf{d}^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla \mathbf{d}^\varepsilon dt = \left(\Delta \mathbf{d}^\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon) + \frac{1}{2}(\mathbf{d}^\varepsilon \times \mathbf{h})\times \mathbf{h}\right) dt + (\mathbf{d}^\varepsilon\times\mathbf{h})dW_2. \qquad (9)$$

Now we apply the Itô formula to $\Phi_\varepsilon(\mathbf{d}^\varepsilon) := \frac{1}{2}\|\nabla \mathbf{d}^\varepsilon\|^2 + \int_D F_\varepsilon(\mathbf{d}^\varepsilon)dx$ (see Appendix A) to get

$$\begin{aligned}
&\Phi_\varepsilon(\mathbf{d}^\varepsilon)(t) - \Phi_\varepsilon(\mathbf{d}_0^\varepsilon) \qquad (10)\\
&= \int_0^t \int_D \langle \mathbf{u}^\varepsilon \cdot \nabla \mathbf{d}^\varepsilon, \Delta \mathbf{d}^\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)\rangle dxds - \int_0^t \int_D |\Delta \mathbf{d}^\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)|^2 dxds\\
&+ \frac{1}{2}\int_0^t \int_D (\langle \nabla \mathbf{d}^\varepsilon, \nabla((\mathbf{d}^\varepsilon\times\mathbf{h})\times\mathbf{h})\rangle + |\nabla(\mathbf{d}^\varepsilon\times\mathbf{h})|^2) dxds\\
&+ \frac{1}{2}\int_0^t \int_D \langle -\Delta \mathbf{d}^\varepsilon + \mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon), \mathbf{d}^\varepsilon \times \mathbf{h}\rangle dxdW_2(s).
\end{aligned}$$

Using the fact that

$$\int_0^t \int_D \langle \mathbf{u}^\varepsilon \cdot \nabla \mathbf{d}^\varepsilon, \mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)\rangle dxds = \int_0^t \int_D \mathbf{u}^\varepsilon \cdot \nabla F_\varepsilon(\mathbf{d}^\varepsilon) dxds = 0,$$

and

$$\begin{aligned}
&- \int_0^t \int_D \langle \mathbf{u}^\varepsilon \cdot \nabla \mathbf{d}^\varepsilon, \Delta \mathbf{d}^\varepsilon \rangle dxds\\
&= \int_0^t \int_D \langle \nabla \mathbf{d}^\varepsilon \odot \nabla \mathbf{d}^\varepsilon, \nabla \mathbf{u}^\varepsilon \rangle dxds + \int_0^t \int_D \mathbf{u}^\varepsilon \cdot \nabla \left(\frac{|\nabla \mathbf{d}^\varepsilon|^2}{2}\right) dxds\\
&= \int_0^t \int_D \langle \nabla \mathbf{d}^\varepsilon \odot \nabla \mathbf{d}^\varepsilon, \nabla \mathbf{u}^\varepsilon \rangle dxds,
\end{aligned}$$

we can add (8) and (10) together to obtain

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{u}^\varepsilon(t)\|^2 + \frac{1}{2}\|\nabla \mathbf{d}^\varepsilon(t)\|^2 \qquad (11)\\
&+ \int_{D\times\{t\}} F_\varepsilon(\mathbf{d}^\varepsilon)dx + \int_0^t \int_D (|\nabla \mathbf{u}^\varepsilon|^2 + |\Delta \mathbf{d}^\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)|^2) dxds\\
&= \frac{1}{2}\|\mathbf{u}_0^\varepsilon\|^2 + \frac{1}{2}\|\nabla \mathbf{d}_0^\varepsilon\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^t (\|S(\mathbf{u}^\varepsilon)\|^2_{\mathcal{L}_2(K_1,\mathbf{H})}+\|\nabla(\mathbf{d}^\varepsilon\times\mathbf{h})\|^2)ds\\
&+\frac{1}{2}\int_0^t\int_D\langle\nabla\mathbf{d}^\varepsilon,\nabla((\mathbf{d}^\varepsilon\times\mathbf{h})\times\mathbf{h})\rangle dxds\\
&+\int_0^t\int_D\langle\mathbf{u}^\varepsilon,S(\mathbf{u}^\varepsilon)dW_1(s)\rangle dx\\
&+\int_0^t\int_D\langle\mathbf{d}^\varepsilon\times\mathbf{h},\Delta\mathbf{d}^\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)\rangle dxdW_2(s).
\end{aligned}
$$

It has been shown in [6, Theorem 5.1] that $\mathbf{d}^\varepsilon$ satisfies the maximum principle, i.e., $|\mathbf{d}^\varepsilon|\le 1$ for almost all $(\omega,t,x)\in\Omega\times[0,T]\times D$ provided $|\mathbf{d}_0^\varepsilon|\le 1$. Hence we have that

$$
\begin{aligned}
&\int_0^t\|S(\mathbf{u}^\varepsilon)\|^2_{\mathcal{L}_2(K_1,\mathbf{H})}ds\le\int_0^t\|S(\mathbf{u}^\varepsilon)\|^2_{\mathcal{L}_2(K_1,\mathbf{J})}ds\le C\int_0^t\int_D(1+|\mathbf{u}^\varepsilon|^2)dxds,\\
&\int_0^t\int_D|\nabla(\mathbf{d}^\varepsilon\times\mathbf{h})|^2dxds\le C\int_0^t\int_D(|\nabla\mathbf{d}^\varepsilon|^2+|\nabla\mathbf{h}|^2)dxds,\\
&\int_0^t\int_D\langle\nabla\mathbf{d}^\varepsilon,\nabla((\mathbf{d}^\varepsilon\times\mathbf{h})\times\mathbf{h})\rangle dxds\le C\int_0^t\int_D(|\nabla\mathbf{d}^\varepsilon|^2+|\nabla\mathbf{h}|^2)dxds.
\end{aligned}
$$

Combine all these estimates above, we arrive at

$$
\begin{aligned}
&\frac{1}{2}\|\mathbf{u}^\varepsilon(t)\|^2+\frac{1}{2}\|\nabla\mathbf{d}^\varepsilon(t)\|^2\\
&+\int_{D\times\{t\}}F_\varepsilon(\mathbf{d}^\varepsilon)dx+\int_0^t\int_D(|\nabla\mathbf{u}^\varepsilon|^2+|\Delta\mathbf{d}^\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)|^2)dxds\\
&\le\frac{1}{2}\|\mathbf{u}_0^\varepsilon\|^2+\frac{1}{2}\|\nabla\mathbf{d}_0^\varepsilon\|^2\\
&+C\int_0^t\int_D(|\mathbf{u}^\varepsilon|^2+|\nabla\mathbf{d}^\varepsilon|^2+|\nabla\mathbf{h}|^2)dxds\\
&+\int_0^t\int_D\langle\mathbf{u}^\varepsilon,S(\mathbf{u}^\varepsilon)dW_1(s)\rangle dx+\int_0^t\int_D\langle\mathbf{d}^\varepsilon\times\mathbf{h},\Delta\mathbf{d}^\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)\rangle dxdW_2(s).
\end{aligned}
\tag{12}
$$

We can derive from taking the expectation of (12) that

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\left[\|\mathbf{u}^\varepsilon(t)\|^2+\|\nabla\mathbf{d}^\varepsilon(t)\|^2+\int_{D\times\{t\}}F_\varepsilon(\mathbf{d}^\varepsilon)dx\right]\\
&+\mathbb{E}\int_0^T\int_D(|\nabla\mathbf{u}^\varepsilon|^2+|\Delta\mathbf{d}^\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)|^2)dxds\\
&\le C\mathbb{E}\int_0^T\int_D(|\mathbf{u}^\varepsilon|^2+|\nabla\mathbf{d}^\varepsilon|^2+|\nabla\mathbf{h}|^2)dxds\\
&+C\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_D\langle\mathbf{u}^\varepsilon(s),S(\mathbf{u}^\varepsilon(s))dW_1(s)\rangle dx\right|\\
&+C\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_D\langle\mathbf{d}^\varepsilon\times\mathbf{h},\Delta\mathbf{d}^\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)\rangle dxdW_2(s)\right|\\
&+C(1+\|\mathbf{u}_0\|^2+\|\nabla\mathbf{d}_0\|^2).
\end{aligned}
\tag{13}
$$

Now we use the Burkholder–Davis–Gundy inequality, Cauchy–Schwarz inequality and Hölder inequality to show that

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_D\langle \mathbf{d}^\varepsilon\times\mathbf{h},\Delta\mathbf{d}^\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)\rangle dxdW_2(s)\right| \qquad (14)\\
&\le C\mathbb{E}\left[\int_0^T\left|\int_D\langle \mathbf{d}^\varepsilon\times\mathbf{h},\Delta\mathbf{d}^\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)\rangle dx\right|^2ds\right]^{\frac12}\\
&\le C\mathbb{E}\left[\int_0^T\|\langle \mathbf{d}^\varepsilon\times\mathbf{h},\Delta\mathbf{d}^\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)\rangle\|^2ds\right]^{\frac12}\\
&\le C\mathbb{E}\left[\sup_{0\le t\le T}\|\mathbf{d}^\varepsilon\times\mathbf{h}\|_{L^\infty(D)}\left(\int_0^T\int_D|\Delta\mathbf{d}^\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)|^2dxds\right)^{\frac12}\right]\\
&\le C\mathbb{E}\sup_{0\le t\le T}\|\mathbf{d}^\varepsilon\times\mathbf{h}\|^2_{L^\infty(D)}+\frac14\mathbb{E}\int_0^T\int_D|\Delta\mathbf{d}^\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)|^2dxds\\
&\le C(\|\mathbf{h}\|_{L^\infty},T,D)+\frac14\mathbb{E}\int_0^T\int_D|\Delta\mathbf{d}^\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)|^2dxds.
\end{aligned}$$

Similarly, we can show

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_D\langle \mathbf{u}^\varepsilon(s),S(\mathbf{u}^\varepsilon(s))\rangle dxdW_1(s)\right| \qquad (15)\\
&\le\frac14\mathbb{E}\sup_{0\le t\le T}\|\mathbf{u}^\varepsilon(t)\|^2+C\mathbb{E}\int_0^T\int_D|\mathbf{u}^\varepsilon|^2dxds.
\end{aligned}$$

Now we can substitute (14) and (15) into (13) to get

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\left[\|\mathbf{u}^\varepsilon\|^2+\|\nabla\mathbf{d}^\varepsilon\|^2+\int_{D\times\{t\}}F_\varepsilon(\mathbf{d}^\varepsilon)dx\right]\\
&+\mathbb{E}\int_0^T\int_D(|\nabla\mathbf{u}^\varepsilon|^2+|\Delta\mathbf{d}^\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)|^2)dxds\\
&\le C\mathbb{E}\int_0^T\int_D(|\mathbf{u}^\varepsilon|^2+|\nabla\mathbf{d}^\varepsilon|^2+|\nabla\mathbf{h}|^2)dxds+C(\|(\mathbf{u}_0,\nabla\mathbf{d}_0)\|,\|\mathbf{h}\|_{L^\infty},T,D).
\end{aligned}$$

It follows from Gronwall's lemma that

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\left[\|\mathbf{u}^\varepsilon(t)\|^2+\|\nabla\mathbf{d}^\varepsilon(t)\|^2+\int_{D\times\{t\}}F_\varepsilon(\mathbf{d}^\varepsilon)dx\right] \qquad (16)\\
&+\mathbb{E}\int_0^T\int_D(|\nabla\mathbf{u}^\varepsilon|^2+|\Delta\mathbf{d}^\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)|^2)dxds\\
&\le C(\|(\mathbf{u}_0,\nabla\mathbf{d}_0)\|,\|\mathbf{h}\|_{L^\infty},\|\nabla\mathbf{h}\|,T,D).
\end{aligned}$$

Furthermore, if we raise both sides of (12) to the power p ($p>1$) and take the expectation, we arrive at

$$\mathbb{E}\sup_{0\le t\le T}\left[\|\mathbf{u}^\varepsilon(t)\|^2+\|\nabla\mathbf{d}^\varepsilon(t)\|^2+\int_{D\times\{t\}}F_\varepsilon(\mathbf{d}^\varepsilon)dx\right]^p \qquad (17)$$

$$
\begin{aligned}
&+\mathbb{E}\left[\int_0^t\int_D(|\nabla\mathbf{u}^\varepsilon|^2+|\Delta\mathbf{d}^\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)|^2)dxds\right]^p\\
&\le C(\|\mathbf{u}_0\|,\|\nabla\mathbf{d}_0\|,p)+CT\mathbb{E}\big(\int_0^T\left[\|\mathbf{u}^\varepsilon(t)\|^2+\|\nabla\mathbf{d}^\varepsilon(t)\|^2+\|\nabla\mathbf{h}\|^2\right]dt\big)^p\\
&+C\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_D\langle\mathbf{u}^\varepsilon(s),S(\mathbf{u}^\varepsilon(s))\rangle dxdW_1(s)\right|^p\\
&+C\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_D\langle\mathbf{d}^\varepsilon\times\mathbf{h},\Delta\mathbf{d}^\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)\rangle dxdW_2(s)\right|^p.
\end{aligned}
$$

Now we apply the Burkholder–Davis–Gundy, Cauchy–Schwarz, and Hölder inequalities to the last two terms in the righg hand side to get

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_D\langle\mathbf{u}^\varepsilon(s),S(\mathbf{u}^\varepsilon(s))dW_1(s)\rangle dx\right|^p\\
&\le C\mathbb{E}\left[\int_0^T\|\mathbf{u}^\varepsilon(s)\|^2\|S(\mathbf{u}^\varepsilon(s))\|^2ds\right]^{\frac{p}{2}}\\
&\le C\mathbb{E}\left[\sup_{0\le t\le T}\|\mathbf{u}^\varepsilon(t)\|^p\left(\int_0^T(1+\|\mathbf{u}^\varepsilon(s)\|^2)ds\right)^{\frac{p}{2}}\right]\\
&\le\frac{1}{4}\mathbb{E}\sup_{0\le t\le T}\|\mathbf{u}^\varepsilon(t)\|^{2p}+C\mathbb{E}\int_0^T(1+\|\mathbf{u}^\varepsilon(s)\|^2)^pds.
\end{aligned}\tag{18}
$$

A similar argument yields

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_D\langle\mathbf{d}^\varepsilon\times\mathbf{h},\Delta\mathbf{d}^\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)\rangle dxdW_2(s)\right|^p\\
&\le\frac{1}{4}\mathbb{E}\left[\int_0^T\int_D|\Delta\mathbf{d}^\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)|^2dxds\right]^p+C\mathbb{E}\sup_{0\le t\le T}\|\mathbf{d}^\varepsilon\times\mathbf{h}\|^{2p}.
\end{aligned}\tag{19}
$$

Combine (17), (18) and (19), by Gronwall's inequality we obtain that for $p\ge 1$, it holds

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\left[\|\mathbf{u}^\varepsilon(t)\|^2+\|\nabla\mathbf{d}^\varepsilon(t)\|^2+\int_{D\times\{t\}}F_\varepsilon(\mathbf{d}^\varepsilon)dx\right]^p\\
&+\mathbb{E}\left[\int_0^T\int_D(|\nabla\mathbf{u}^\varepsilon|^2+|\Delta\mathbf{d}^\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)|^2)dxds\right]^p\\
&\le C(\|(\mathbf{u}_0,\nabla\mathbf{d}_0)\|,\|\mathbf{h}\|_{L^\infty},\|\nabla\mathbf{h}\|,T,D,p).
\end{aligned}\tag{20}
$$

Similar to the Aubin-Lions lemma in the deterministic case, we need some fractional Sobolev estimates in t variable as in [13] for stochastic Navier-Stokes equations. Write

$$
\begin{aligned}
\mathbf{u}^\varepsilon(t)=\mathbf{u}_0^\varepsilon&+\int_0^t\mathbf{P}\Delta\mathbf{u}^\varepsilon(s)ds-\int_0^t\mathbf{P}\nabla\cdot(\mathbf{u}^\varepsilon\otimes\mathbf{u}^\varepsilon)(s)ds\\
&-\int_0^t\mathbf{P}\nabla\cdot(\nabla\mathbf{d}^\varepsilon\odot\nabla\mathbf{d}^\varepsilon)(s)ds+\int_0^t S(\mathbf{u}^\varepsilon(s))dW_1(s)
\end{aligned}
$$

$$:= \mathbf{u}_0^\varepsilon + \sum_{i=1}^4 I_i^\varepsilon(t),$$

where $\mathbf{P}$ is the Leray projection operator. We have that

$$\mathbb{E}\left[\|I_1^\varepsilon\|^2_{W^{1,2}([0,T];H^{-1}(D))} + \|I_2^\varepsilon\|^2_{W^{1,2}([0,T];H^{-1}(D))}\right] \le C,$$
$$\mathbb{E}\left[\|I_3^\varepsilon\|^2_{W^{1,2}([0,T];W^{-2,\tilde{p}}(D))}\right] \le C, \text{ for some } \tilde{p} > 2.$$

Applying [13, Lemma 2.1] to I_4^ε we conclude that for any $\alpha \in (0, \frac{1}{2})$ and $p \in [2, \infty)$, it holds

$$\begin{aligned}\mathbb{E}[\|I_4^\varepsilon\|^p_{W^{\alpha,p}([0,T];L^2(D))}] &= \mathbb{E}\left\|\int_0^t S(\mathbf{u}^\varepsilon(s))dW_1(s)\right\|^p_{W^{\alpha,p}([0,T];L^2(D))} \\ &\le C\mathbb{E}\int_0^T \|S(\mathbf{u}^\varepsilon(t))\|^p_{\mathcal{L}_2(K_1,\mathbf{H})}dt \\ &\le C\mathbb{E}\int_0^T (1 + \|\mathbf{u}^\varepsilon(t)\|^p_{L^2(D)})dt \le C.\end{aligned}$$

Now we define

$$\begin{aligned}X := &L^\infty([0,T];L^2(D)) \cap L^2([0,T];H^1(D)) \\ &\cap \left(W^{1,2}([0,T];H^{-1}(D)) + W^{1,2}([0,T];W^{-2,\tilde{p}}(D)) + W^{\alpha,p}([0,T];L^2(D))\right).\end{aligned}$$

Let $\{\mathcal{L}(\mathbf{u}^\varepsilon)\}_{0<\varepsilon<1}$ be a family of probability measures define on X as following:

$$\mathcal{L}(\mathbf{u}^\varepsilon)(B) := \mathbb{P}(\mathbf{u}^\varepsilon \in B)$$

for any Borel set $B \subset X$. For a fix $R > 0$, we can derive from Chebyshev's inequality that

$$\begin{aligned}&\mathbb{P}(\|\mathbf{u}^\varepsilon\|_X > R) \\ &\le \mathbb{P}\left(\|\mathbf{u}^\varepsilon\|_{L^\infty([0,T];L^2(D))} > \frac{R}{3}\right) + \mathbb{P}\left(\|\mathbf{u}^\varepsilon\|_{L^2([0,T];H^1(D))} > \frac{R}{3}\right) \\ &+ \mathbb{P}\left(\|\mathbf{u}^\varepsilon\|_{W^{1,2}([0,T];H^{-1}(D))+W^{1,2}([0,T];W^{-2,p}(D))+W^{\alpha,p}([0,T];L^2(D))} > \frac{R}{3}\right) \\ &\le \frac{C}{R}.\end{aligned}$$

By a fractional version of Aubin-Lions lemma and the Sobolev interpolation inequality, X is compactly embedded in $L^p([0,T];L^p(D)) \cap C([0,T];W^{-2,\tilde{p}}(D))$ for $1 < p < 4$ (c.f. [13, 26]). Therefore $\{\mathcal{L}(\mathbf{u}^\varepsilon)\}_{0<\varepsilon<1}$ is tight in $L^p([0,T];L^p(D)) \cap C([0,T];W^{-2,\tilde{p}}(D))$ for $1 < p < 4$. Similarly, we have

$$\begin{aligned}\mathbf{d}^\varepsilon(t) = \mathbf{d}_0^\varepsilon &- \int_0^t \nabla\cdot(\mathbf{u}^\varepsilon \otimes \mathbf{d}^\varepsilon)(s)ds + \int_0^t (\Delta\mathbf{d}^\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon))(s)ds \\ &+ \frac{1}{2}\int_0^t ((\mathbf{d}^\varepsilon \times \mathbf{h}) \times \mathbf{h})(s)ds + \int_0^t (\mathbf{d}^\varepsilon \times \mathbf{h})(s)dW_2(s) \\ &:= \mathbf{d}_0^\varepsilon + \sum_{i=1}^4 J_i^\varepsilon(t).\end{aligned}$$

Then we have

$$\mathbb{E}\left[\|J_1^\varepsilon\|^{\frac{4}{3}}_{W^{1,\frac{4}{3}}([0,T];L^{\frac{4}{3}}(D))}\right] < C,$$
$$\mathbb{E}\left[\|J_2^\varepsilon\|^2_{W^{1,2}([0,T];L^2(D))} + \|J_3^\varepsilon\|^2_{W^{1,2}([0,T];L^\infty(D))}\right] \le C,$$

and by an argument similar to that of I_4^ε we can show that for any $\alpha \in (0, \frac{1}{2})$ and $p \in [2, \infty)$, it holds

$$\begin{aligned}\mathbb{E}\left[\|J_4^\varepsilon\|^p_{W^{\alpha,p}([0,T];L^2(D))}\right] &= \mathbb{E}\left\|\int_0^t \mathbf{d}^\varepsilon \times \mathbf{h} dW_2(s)\right\|^p_{W^{\alpha,p}([0,T];L^2(D))} \\ &\le C\mathbb{E}\int_0^T \|\mathbf{d}^\varepsilon \times \mathbf{h}(t)\|^p_{L^2(D)}\, dt \\ &\le C\mathbb{E}\int_0^T \|\mathbf{h}\|^p_{L^\infty}\, \|\mathbf{d}^\varepsilon(t)\|^p_{L^2(D)} dt \le C.\end{aligned}$$

Hence, the laws $\{\mathcal{L}(\mathbf{d}^\varepsilon)\}_{0<\varepsilon<1}$ are bounded in probability in

$$\begin{aligned}&Y := L^\infty([0,T]; H^1(D)) \\ &\cap \left(W^{1,\frac{4}{3}}([0,T]; L^{\frac{4}{3}}(D)) + W^{1,2}([0,T]; L^2(D)) + W^{\alpha,p}([0,T]; L^2(D))\right).\end{aligned}$$

Since Y is compactly embedded into $L^q([0,T]; L^q(D)) \cap C([0,T]; L^{\frac{4}{3}}(D))$, $p > 1$, $\{\mathcal{L}(\mathbf{d}^\varepsilon)\}_{0<\varepsilon<1}$ is tight in $L^q([0,T]; L^q(D)) \cap C([0,T]; L^{\frac{4}{3}}(D))$, $p > 1$.

3. **Convergence of Ginzburg-Landau approximation.** The main purpose of this section is mainly devoted to show the convergence of Ericksen stress tensor and the martingale terms. From the uniform energy estimates in the previous section, we know that $(\mathcal{L}(\mathbf{u}^\varepsilon), \mathcal{L}(\mathbf{d}^\varepsilon))$ is tight in $L^p([0,T]; L^p(D)) \cap C([0,T]; W^{-2,\tilde{p}}(D)) \times L^q([0,T]; L^q(D)) \cap C([0,T]; L^{\frac{4}{3}}(D))$ for $1 < p < 4, 1 < q < \infty$. Now we apply the Prohorov's theorem, there exists a probability measure μ on $L^p([0,T]; L^p(D)) \cap C([0,T]; W^{-2,\tilde{p}}(D)) \times L^q([0,T]; L^q(D)) \cap C([0,T]; L^{\frac{4}{3}}(D)) \times C([0,T]; K_2) \times C([0,T])$, $1 < p < 4, 1 < q < \infty$ such that after passing to a subsequence,

$$\mathcal{L}(\mathbf{u}^\varepsilon, \mathbf{d}^\varepsilon, W_1, W_2) \rightharpoonup \mu.$$

Then by Skorokhod's embedding theorem, there exists a complete probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and a sequence of random variables $(\overline{\mathbf{u}}^\varepsilon, \overline{\mathbf{d}}^\varepsilon, \overline{W}_1^\varepsilon, \overline{W}_2^\varepsilon)$ on $(\Omega', \mathcal{F}', \mathbb{P}')$ such that

$$\mathcal{L}(\overline{\mathbf{u}}^\varepsilon, \overline{\mathbf{d}}^\varepsilon, \overline{W}_1^\varepsilon, \overline{W}_2^\varepsilon) = \mathcal{L}(\mathbf{u}^\varepsilon, \mathbf{d}^\varepsilon, W_1, W_2), \tag{21}$$

and $(\mathbf{u}, \mathbf{d}, W_1', W_2')$ defined on $(\Omega', \mathcal{F}', \mathbb{P}')$ such that

$$\begin{cases} \mathcal{L}(\mathbf{u}, \mathbf{d}, W_1', W_2') = \mu, \\ \overline{\mathbf{u}}^\varepsilon \to \mathbf{u} \text{ in } L^p([0,T]; L^p(D)) \cap C([0,T]; W^{-2,\tilde{p}}(D)), 1 < p < 4, \mathbb{P}'\text{-a.s.}, \\ \overline{\mathbf{u}}^\varepsilon \rightharpoonup \mathbf{u} \text{ in } L^2(\Omega' \times [0,T]; \mathbf{J}), \\ \overline{\mathbf{d}}^\varepsilon \to \mathbf{d} \text{ in } L^q([0,T]; L^q(D)) \cap C([0,T]; L^{\frac{4}{3}}(D)), 1 < q < \infty, \mathbb{P}'\text{-a.s.}, \\ \overline{\mathbf{d}}^\varepsilon \rightharpoonup \mathbf{d} \text{ in } L^2(\Omega' \times [0,T], H^1), \\ W_1^\varepsilon \to W_1' \text{ in } C([0,T]; K_2), \quad \mathbb{P}'\text{-a.s.}, \\ W_2^\varepsilon \to W_2' \text{ in } C([0,T]; \mathbb{R}), \quad \mathbb{P}'\text{-a.s.}. \end{cases} \tag{22}$$

And for $\mathbb{P}'$-a.s., $\mathbf{u} \in L^\infty([0,T]; \mathbf{H}) \cap L^2([0,T]; \mathbf{J})$, $\mathbf{d} \in L^\infty([0,T]; H^1(D))$.

For martingale solutions, for each $0<\varepsilon<1$, we define $M_{\mathbf{u}^\varepsilon}(t), M_{\mathbf{d}^\varepsilon}(t)$ as

$$M_{\mathbf{u}^\varepsilon}(t)=\mathbf{u}^\varepsilon(t)-\mathbf{u}_0^\varepsilon+\int_0^t[\mathbf{P}\nabla\cdot(\mathbf{u}^\varepsilon\otimes\mathbf{u}^\varepsilon)-\mathbf{P}\Delta\mathbf{u}^\varepsilon+\mathbf{P}\nabla\cdot(\nabla\mathbf{d}^\varepsilon\odot\nabla\mathbf{d}^\varepsilon)](s)ds,$$

$$M_{\mathbf{d}^\varepsilon}(t)=\mathbf{d}^\varepsilon(t)-\mathbf{d}_0^\varepsilon+\int_0^t[\nabla\cdot(\mathbf{u}^\varepsilon\otimes\mathbf{d}^\varepsilon)-\Delta\mathbf{d}^\varepsilon+\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)-\frac{1}{2}(\mathbf{d}^\varepsilon\times\mathbf{h})\times\mathbf{h}](s)ds,$$

for any $t\in(0,T]$. Also define $M_{\overline{\mathbf{u}}^\varepsilon}, M_{\overline{\mathbf{d}}^\varepsilon}$ by replacing $\mathbf{u}^\varepsilon,\mathbf{d}^\varepsilon$ in $M_{\mathbf{u}^\varepsilon}, M_{\mathbf{d}^\varepsilon}$ by $\overline{\mathbf{u}}^\varepsilon,\overline{\mathbf{d}}^\varepsilon$.

Next we show that for $\mathbb{P}'$-a.s.,

$$M_{\overline{\mathbf{u}}^\varepsilon}(t)=\int_0^t S(\overline{\mathbf{u}}^\varepsilon)d\overline{W}_1^\varepsilon(s), \tag{23}$$

$$M_{\overline{\mathbf{d}}^\varepsilon}(t)=\int_0^t(\overline{\mathbf{d}}^\varepsilon\times\mathbf{h})d\overline{W}_2^\varepsilon(s) \tag{24}$$

for every $\varepsilon>0$ and every $t\in[0,T]$. For any $\mathbf{z}\in L^2(0,T;H^{-1})$ we set

$$\varphi(\mathbf{z})=\frac{\int_0^T\|\mathbf{z}(s)\|_{H^{-1}}^2ds}{1+\int_0^T\|\mathbf{z}(s)\|_{H^{-1}}^2ds}.$$

By a argument similar to that in [1, 6] we can show that

$$\mathbb{E}'\varphi\left(M_{\overline{\mathbf{u}}^\varepsilon}(\cdot)-\int_0^\cdot S(\overline{\mathbf{u}}^\varepsilon(s))d\overline{W}_1^\varepsilon(s)\right)=\mathbb{E}\varphi\left(M_{\mathbf{u}^\varepsilon}(\cdot)-\int_0^\cdot S(\mathbf{u}^\varepsilon(s))dW_1(s)\right)=0.$$

This implies that for $\mathbb{P}'$-a.s. (23) holds for all $t\in(0,T]$. Similarly, we can show (24) is also true.

Let $M_{\mathbf{u}}(t)$ and $M_{\mathbf{d}}(t)$ be defined by

$$M_{\mathbf{u}}(t)=\mathbf{u}(t)-\mathbf{u}_0+\int_0^t[\mathbf{P}\nabla\cdot(\mathbf{u}\otimes\mathbf{u})-\mathbf{P}\Delta\mathbf{u}+\mathbf{P}\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d})](s)ds,$$

$$M_{\mathbf{d}}(t)=\mathbf{d}(t)-\mathbf{d}_0+\int_0^t[\nabla\cdot(\mathbf{u}\otimes\mathbf{d})-\Delta\mathbf{d}-|\nabla\mathbf{d}|^2\mathbf{d}-\frac{1}{2}(\mathbf{d}\times\mathbf{h})\times\mathbf{h}](s)ds.$$

With (22), we have the almost surely convergence of every term in $M_{\overline{\mathbf{u}}^\varepsilon}$ except the Ericksen stress tensor $(\nabla\overline{\mathbf{d}}^\varepsilon\odot\nabla\overline{\mathbf{d}}^\varepsilon)$. Now we claim that for $\mathbb{P}'$-a.s.

$$\begin{aligned}&\lim_{\varepsilon\to0}\int_0^T\int_D\langle\nabla\overline{\mathbf{d}}^\varepsilon\otimes\nabla\overline{\mathbf{d}}^\varepsilon-\frac{1}{2}|\nabla\overline{\mathbf{d}}^\varepsilon|^2\mathbb{I}_2,\nabla\varphi\rangle dxds\\&=\int_0^T\int_D\langle\nabla\mathbf{d}\otimes\nabla\mathbf{d}-\frac{1}{2}|\nabla\mathbf{d}|^2\mathbb{I}_2,\nabla\varphi\rangle dxds.\end{aligned} \tag{25}$$

For any $0<\Lambda_1,\Lambda_2<\infty$, define the set $\mathbf{X}(\Lambda_1,\Lambda_2)$ consisting of solutions $\overline{\mathbf{d}}^\varepsilon$ to

$$\Delta\overline{\mathbf{d}}^\varepsilon-\mathbf{f}_\varepsilon(\overline{\mathbf{d}}^\varepsilon)=\tau^\varepsilon \text{ in } D \tag{26}$$

such that the following properties hold:

1. $|\overline{\mathbf{d}}^\varepsilon|\le1$ for a.e. $x\in D$.
2. $$\sup_{0<\varepsilon\le1}\mathcal{E}_\varepsilon(\overline{\mathbf{d}}^\varepsilon)=\int_D\left(\frac{1}{2}|\nabla\overline{\mathbf{d}}^\varepsilon|^2+F_\varepsilon(\overline{\mathbf{d}}^\varepsilon)\right)dx\le\Lambda_1.$$
3. $$\sup_{0<\varepsilon\le1}\|\tau^\varepsilon\|_{L^2(D)}\le\Lambda_2.$$

The following small energy regularity lemma [18, 25] plays a key role in our analysis.

Lemma 3.1. *Suppose* $\{\overline{\mathbf{d}}^\varepsilon\}_{0<\varepsilon\le 1} \subset \mathbf{X}(\Lambda_1, \Lambda_2)$ *and* $\tau^\varepsilon \rightharpoonup \tau$ *in* $L^2(D)$. *Then there exists a* $\delta_0 > 0$ *such that if for* $x_0 \in D$ *and* $0 < r_0 < \operatorname{dist}(x_0, \partial\Omega)$,

$$\sup_{0<\varepsilon\le 1} \int_{B_{r_0}(x_0)} \left(\frac{1}{2}|\nabla \overline{\mathbf{d}}^\varepsilon|^2 + F_\varepsilon(\overline{\mathbf{d}}^\varepsilon)\right) dx \le \delta_0^2, \tag{27}$$

then there exists an approximated harmonic map $\mathbf{d} \in H^1(B_{\frac{r_0}{4}}(x_0), \mathbb{S}^2)$ *with tensor field* τ, *i.e.,*

$$\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d} = \tau, \tag{28}$$

such that

$$\overline{\mathbf{d}}^\varepsilon \to \mathbf{d} \text{ in } H^1(B_{\frac{r_0}{4}}(x_0)) \tag{29}$$

as $\varepsilon \to 0$.

This leads to the following H^1 precompactness result.

Lemma 3.2. *Under the same assumption as Lemma 3.1,*

$$\overline{\mathbf{d}}^\varepsilon \to \mathbf{d} \text{ in } H^1_{loc}(D \setminus \Sigma),$$

where

$$\Sigma := \bigcap_{r>0} \left\{ x \in D : \liminf_{\varepsilon\to 0} \int_{B_r(x)} \left(\frac{1}{2}|\nabla \overline{\mathbf{d}}^\varepsilon|^2 + F_\varepsilon(\overline{\mathbf{d}}^\varepsilon)\right) dx > \delta_0^2 \right\}.$$

Moreover, Σ *is a finite set.*

From (16) and (21), we have

$$\begin{aligned}
&\mathbb{E}' \sup_{0\le t\le T} \left[\|\overline{\mathbf{u}}^\varepsilon(t)\|^2 + \|\nabla \overline{\mathbf{d}}^\varepsilon(t)\|^2 + \int_{D\times\{t\}} F_\varepsilon(\overline{\mathbf{d}}^\varepsilon) dx \right] \\
&+ \mathbb{E}' \left[\int_0^T (\|\nabla \overline{\mathbf{d}}^\varepsilon\|^2 + \|\Delta \overline{\mathbf{d}}^\varepsilon - \mathbf{f}_\varepsilon(\overline{\mathbf{d}}^\varepsilon)\|^2) dt \right] \\
&= \mathbb{E} \sup_{0\le t\le T} \left[\|\mathbf{u}^\varepsilon(t)\|^2 + \|\nabla \mathbf{d}^\varepsilon(t)\|^2 + \int_{D\times\{t\}} F_\varepsilon(\mathbf{d}^\varepsilon) dx \right] \\
&+ \mathbb{E} \left[\int_0^T (\|\nabla \mathbf{d}^\varepsilon\|^2 + \|\Delta \mathbf{d}^\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)\|^2) dt \right] \\
&\le C.
\end{aligned} \tag{30}$$

Hence, there exists $\mathcal{N} \subset \Omega'$ such that $\mathbb{P}'(\mathcal{N}) = 0$, and it holds for $\omega \in \Omega' \setminus \mathcal{N}$ that

$$\liminf_{\varepsilon\to 0} \int_0^T \int_D (|\nabla \overline{\mathbf{d}}^\varepsilon|^2 + |\Delta \overline{\mathbf{d}}^\varepsilon - \mathbf{f}_\varepsilon(\overline{\mathbf{d}}^\varepsilon)|^2) dx dt = C_1(\omega) < \infty, \tag{31}$$

and

$$\liminf_{\varepsilon\to 0} \sup_{0\le t\le T} \int_{D\times\{t\}} (|\overline{\mathbf{u}}^\varepsilon|^2 + |\nabla \overline{\mathbf{d}}^\varepsilon|^2 + F_\varepsilon(\overline{\mathbf{d}}^\varepsilon)) dx = C_2(\omega) < \infty. \tag{32}$$

Now fix $\omega \in \Omega' \setminus \mathcal{N}$, by Fatou's lemma, we have

$$\begin{aligned}
&\int_0^T \liminf_{\varepsilon\to 0} \int_D (|\nabla \overline{\mathbf{d}}^\varepsilon|^2 + |\Delta \overline{\mathbf{d}}^\varepsilon - \mathbf{f}_\varepsilon(\overline{\mathbf{d}}^\varepsilon)|^2) dx ds \\
&\le \liminf_{\varepsilon\to 0} \int_0^T \int_D (|\nabla \overline{\mathbf{d}}^\varepsilon|^2 + |\Delta \overline{\mathbf{d}}^\varepsilon - \mathbf{f}_\varepsilon(\overline{\mathbf{d}}^\varepsilon)|^2) dx ds < \infty.
\end{aligned}$$

Hence there exists $A \subset [0,T]$ with full Lebesgue such that for any $t \in A$,

$$\liminf_{\varepsilon\to 0} \int_{D\times\{t\}} (|\nabla \overline{\mathbf{d}}^{\varepsilon}|^2 + |\Delta \overline{\mathbf{d}}^{\varepsilon} - \mathbf{f}_\varepsilon(\overline{\mathbf{d}}^{\varepsilon})|^2)dx < \infty.$$

For $t \in A$, we set

$$\Sigma_t := \bigcap_{r>0} \left\{ x \in D : \liminf_{\varepsilon\to 0} \int_{B_r(x)\times\{t\}} (\frac{1}{2}|\nabla \overline{\mathbf{d}}^{\varepsilon}|^2 + F_\varepsilon(\overline{\mathbf{d}}^{\varepsilon}))dx > \delta_0^2 \right\}.$$

By Lemma 3.2, it holds that $\#(\Sigma_t) \le C_3(\omega) < \infty$ and

$$\overline{\mathbf{d}}^{\varepsilon}(t) \to \mathbf{d}(t) \text{ in } H^1_{\text{loc}}(D \setminus \Sigma_t).$$

Hence we get (25) holds for φ with supp $\varphi \subset D \setminus \Sigma_t$. Now we consider the case $\Sigma_t \cap \text{supp } \varphi \neq \emptyset$. Since Σ_t is finite, we may assume $(0,0) \in \text{supp } \varphi$. Write

$$\nabla \overline{\mathbf{d}}^{\varepsilon} \odot \nabla \overline{\mathbf{d}}^{\varepsilon} - \frac{1}{2}|\nabla \overline{\mathbf{d}}^{\varepsilon}|^2 \mathbb{I}_2 = \frac{1}{2}\begin{pmatrix} |\partial_{x_1}\overline{\mathbf{d}}^{\varepsilon}|^2 - |\partial_{x_2}\overline{\mathbf{d}}^{\varepsilon}|^2 & 2\langle \partial_{x_1}\overline{\mathbf{d}}^{\varepsilon}, \partial_{x_2}\overline{\mathbf{d}}^{\varepsilon}\rangle \\ 2\langle \partial_{x_1}\overline{\mathbf{d}}^{\varepsilon}, \partial_{x_2}\overline{\mathbf{d}}^{\varepsilon}\rangle & |\partial_{x_2}\overline{\mathbf{d}}^{\varepsilon}|^2 - |\partial_{x_1}\overline{\mathbf{d}}^{\varepsilon}|^2 \end{pmatrix}. \tag{33}$$

We can now assume that there exists two real number α, β such that

$$\begin{aligned}&\left(\nabla \overline{\mathbf{d}}^{\varepsilon} \odot \nabla \overline{\mathbf{d}}^{\varepsilon} - \frac{1}{2}|\nabla \overline{\mathbf{d}}^{\varepsilon}|^2 \mathbb{I}_2\right) dx \\ &\rightharpoonup \left(\frac{1}{2}\nabla \mathbf{d} \odot \nabla \mathbf{d} - \frac{1}{2}|\nabla \mathbf{d}|^2 \mathbb{I}_2\right) dx + \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \delta_{(0,0)}\end{aligned}$$

as convergence of Radon measures. (25) is true if we can show

$$\alpha = \beta = 0.$$

We apply the same Pohozaev argument as that in [10]. Set τ^ε, e_ε to be

$$\Delta \overline{\mathbf{d}}^{\varepsilon} - \mathbf{f}_\varepsilon(\overline{\mathbf{d}}^{\varepsilon}) =: \tau^\varepsilon \tag{34}$$

and

$$e_\varepsilon(\overline{\mathbf{d}}^{\varepsilon}) := \frac{1}{2}|\nabla \overline{\mathbf{d}}^{\varepsilon}|^2 + F_\varepsilon(\overline{\mathbf{d}}^{\varepsilon}).$$

For any $X \in C^\infty(D, \mathbb{R}^2)$, multiplying (34) by $X \cdot \nabla \overline{\mathbf{d}}^{\varepsilon}$ and integrating over $B_r(0)$ we get

$$\begin{aligned}&\int_{\partial B_r(0)} \langle X \cdot \nabla \overline{\mathbf{d}}^{\varepsilon}, \frac{x}{|x|}\rangle d\sigma - \int_{B_r(0)} \langle \nabla \overline{\mathbf{d}}^{\varepsilon} \odot \nabla \overline{\mathbf{d}}^{\varepsilon}, \nabla X\rangle dx \\ &+ \int_{B_r(0)} \operatorname{div} X e_\varepsilon(\overline{\mathbf{d}}^{\varepsilon})dx - \int_{\partial B_r(0)} e_\varepsilon(\overline{\mathbf{d}}^{\varepsilon})\langle X, \frac{x}{|x|}\rangle d\sigma \\ &= \int_{B_r(0)} \langle X \cdot \nabla \overline{\mathbf{d}}^{\varepsilon}, \tau^\varepsilon\rangle dx.\end{aligned} \tag{35}$$

If we choose $X(x) = x$, then (35) becomes

$$\begin{aligned}&r\int_{\partial B_r(0)} \left|\frac{\partial \overline{\mathbf{d}}^{\varepsilon}}{\partial r}\right|^2 d\sigma + \int_{B_r(0)} 2F_\varepsilon(\overline{\mathbf{d}}^{\varepsilon})dx \\ &- r\int_{\partial B_r(0)} e_\varepsilon(\overline{\mathbf{d}}^{\varepsilon})d\sigma = \int_{B_r(0)} |x|\langle \frac{\partial \overline{\mathbf{d}}^{\varepsilon}}{\partial r}, \tau^\varepsilon\rangle dx.\end{aligned}$$

Hence

$$\int_{\partial B_r(0)} e_\varepsilon(\overline{\mathbf{d}}^\varepsilon)d\sigma = \int_{\partial B_r(0)} \left|\frac{\partial \overline{\mathbf{d}}^\varepsilon}{\partial r}\right| d\sigma$$
$$+ \frac{1}{r}\int_{B_r(0)} 2F_\varepsilon(\overline{\mathbf{d}}^\varepsilon)dx - \frac{1}{r}\int_{B_r(0)} |x|\langle \frac{\partial \overline{\mathbf{d}}^\varepsilon}{\partial r}, \tau^\varepsilon\rangle dx.$$

Integrating from r to R yields

$$\begin{aligned}\int_{B_R(0)\setminus B_r(0)} e_\varepsilon(\overline{\mathbf{d}}^\varepsilon)dx &= \int_{B_R(0)\setminus B_r(0)} \left|\frac{\partial \overline{\mathbf{d}}^\varepsilon}{\partial r}\right|^2 dx \\ &+ \int_r^R \frac{1}{\tau}\int_{B_\tau(0)} \left(2F_\varepsilon(\overline{\mathbf{d}}^\varepsilon) - |x|\langle \frac{\partial \overline{\mathbf{d}}^\varepsilon}{\partial r}, \tau^\varepsilon\rangle\right) dxd\tau.\end{aligned} \tag{36}$$

Since $\Sigma_t = (0,0)$, then there exists $\gamma > 0$ such that

$$e_\varepsilon(\overline{\mathbf{d}}^\varepsilon)dx \rightharpoonup \frac{1}{2}|\nabla \mathbf{d}|^2 dx + \gamma\delta_{(0,0)}$$

as convergence of Radon measure. By sending $\varepsilon \to 0$ in (36) we get

$$\begin{aligned}&\int_{B_R(0)\setminus B_r(0)} \frac{1}{2}|\nabla \mathbf{d}|^2 dx \\ &\geq \int_{B_R(0)\setminus B_r(0)} \left|\frac{\partial \mathbf{d}}{\partial r}\right|^2 dx + \int_r^R \frac{1}{\tau}\liminf_{\varepsilon\to 0}\int_{B_\tau(0)} 2F_\varepsilon(\overline{\mathbf{d}}^\varepsilon)dxd\tau \\ &+ \liminf_{\varepsilon\to 0}\int_r^R -\frac{1}{\tau}\int_{B_\tau(0)} |x|\langle \frac{\partial \overline{\mathbf{d}}^\varepsilon}{\partial r}, \tau^\varepsilon\rangle dxd\tau.\end{aligned} \tag{37}$$

Notice that

$$\begin{aligned}&\left|\int_r^R -\frac{1}{\tau}\liminf_{\varepsilon\to 0}\int_{B_\tau(0)} |x|\langle \frac{\partial \overline{\mathbf{d}}^\varepsilon}{\partial r}, \tau^\varepsilon\rangle dxd\tau\right| \\ &\leq \limsup_{\varepsilon\to 0}\int_0^R \|\tau^\varepsilon\|_{L^2(B_\tau(0))} \left\|\nabla\overline{\mathbf{d}}^\varepsilon\right\|_{L^2(B_\tau(0))} d\tau \\ &= \mathcal{O}(R).\end{aligned}$$

As a consequence, we claim that

$$2F_\varepsilon(\overline{\mathbf{d}}^\varepsilon) \to 0 \text{ in } L^1(B_\delta). \tag{38}$$

For, otherwise, then there exists $\kappa > 0$ such that

$$2F_\varepsilon(\overline{\mathbf{d}}^\varepsilon)dx \rightharpoonup \kappa\delta_{(0,0)}.$$

This implies

$$\lim_{r\downarrow 0}\int_r^R \frac{1}{\tau}\liminf_{\varepsilon\to 0}\int_{B_\tau(0)} 2F_\varepsilon(\overline{\mathbf{d}}^\varepsilon)dxd\tau = \lim_{r\downarrow 0}\int_r^R \frac{\kappa}{\tau}d\tau = \infty.$$

If we choose $X(x) = (x_1, 0)$ in (35), we obtain that

$$\begin{aligned}
&\frac{1}{2}\int_{B_r(0)} \left(|\partial_{x_2}\overline{\mathbf{d}}^\varepsilon|^2 - |\partial_{x_1}\overline{\mathbf{d}}^\varepsilon|^2\right) dx + \int_{B_r(0)} F_\varepsilon(\overline{\mathbf{d}}^\varepsilon) dx \\
&= \int_{B_r(0)} x_1 \langle \partial_{x_1}\overline{\mathbf{d}}^\varepsilon, \tau^\varepsilon\rangle dx + \int_{\partial B_r(0)} \frac{x_1^2}{r} e_\varepsilon(\overline{\mathbf{d}}^\varepsilon) d\sigma \\
&- \int_{\partial B_r(0)} x_1 \langle \partial_{x_1}\overline{\mathbf{d}}^\varepsilon, \frac{\partial \overline{\mathbf{d}}^\varepsilon}{\partial r}\rangle d\sigma.
\end{aligned} \tag{39}$$

Since $e_\varepsilon(\overline{\mathbf{d}}^\varepsilon)dx \rightharpoonup \frac{1}{2}|\nabla \mathbf{d}|^2 dx$ in $B_{2r} \setminus B_{\frac{r}{2}}$ for $r > 0$, it is easy to see

$$\begin{aligned}
&\int_{\partial B_r(0)} x_1 \langle \partial_{x_1}\overline{\mathbf{d}}^\varepsilon, \frac{\partial \overline{\mathbf{d}}^\varepsilon}{\partial r}\rangle d\sigma \to \int_{\partial B_r(0)} x_1 \langle \partial_{x_1}\mathbf{d}, \frac{\partial \mathbf{d}}{\partial r}\rangle d\sigma, \\
&\int_{\partial B_r(0)} \frac{x_1^2}{r} e_\varepsilon(\overline{\mathbf{d}}^\varepsilon) d\sigma \to \frac{1}{2}\int_{\partial B_r} \frac{x_1^2}{r} |\nabla \mathbf{d}|^2 d\sigma,
\end{aligned}$$

and by (38),

$$\int_{B_r(0)} F_\varepsilon(\overline{\mathbf{d}}^\varepsilon) dx \to 0.$$

With the fact that

$$\left|\int_{B_r} x_1 \langle \partial_{x_1}\overline{\mathbf{d}}^\varepsilon, \tau^\varepsilon\rangle dx\right| = \mathcal{O}(r),$$

by sending $\varepsilon \to 0$ in (39) we obtain

$$\frac{1}{2}\int_{B_r(0)} \left(|\partial_{x_2}\mathbf{d}|^2 - |\partial_{x_1}\mathbf{d}|^2\right) dx + \alpha = \mathcal{O}(r)$$

which implies $\alpha = 0$ after sending $r \to 0$.

Similarly, if we choose $X(x) = (0, x_1)$ in (35), by performing the same argument we will arrive at

$$\frac{1}{2}\int_{B_r(0)} \langle \partial_{x_1}\mathbf{d}, \partial_{x_2}\mathbf{d}\rangle dx + \beta = \mathcal{O}(r).$$

Hence $\beta = 0$. This implies almost surely convergence of Ericksen stress tensor field (25). From (20) and (21) we can conclude that for any $1 < p < \infty$, it holds

$$\begin{aligned}
&\mathbb{E}' \sup_{0\le t\le T} \left[\|\overline{\mathbf{u}}^\varepsilon(t)\|^2 + \|\nabla \overline{\mathbf{d}}^\varepsilon(t)\|^2 + \int_{D\times\{t\}} F_\varepsilon(\overline{\mathbf{d}}^\varepsilon) dx\right]^p \\
&+ \mathbb{E}' \left[\int_0^T (\|\nabla \overline{\mathbf{u}}^\varepsilon\|^2 + \|\Delta \overline{\mathbf{d}}^\varepsilon - \mathbf{f}_\varepsilon(\overline{\mathbf{d}}^\varepsilon)\|^2) dt\right]^p \\
&= \mathbb{E} \sup_{0\le t\le T} \left[\|\mathbf{u}^\varepsilon(t)\|^2 + \|\nabla \mathbf{d}^\varepsilon(t)\|^2 + \int_{D\times\{t\}} F_\varepsilon(\mathbf{d}^\varepsilon) dx\right]^p \\
&+ \mathbb{E} \left[\int_0^T (\|\nabla \mathbf{u}^\varepsilon\|^2 + \|\Delta \mathbf{d}^\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)\|^2) dt\right]^p \\
&\le C.
\end{aligned} \tag{40}$$

Thus we have for any $\xi \in L^2(\Omega'; \mathbf{J})$, it holds

$$\lim_{\varepsilon\to 0} \mathbb{E}' \left[\int_D \langle M_{\overline{\mathbf{u}}^\varepsilon}(t), \xi\rangle dx\right] \tag{41}$$

$$
\begin{aligned}
&= \lim_{\varepsilon\to 0}\mathbb{E}'\Big[\int_D \langle \overline{\mathbf{u}}^\varepsilon(t)-\mathbf{u}_0^\varepsilon,\xi\rangle dx \\
&\quad + \int_0^t\int_D (\langle \nabla\overline{\mathbf{u}}^\varepsilon,\nabla\xi\rangle - \langle \overline{\mathbf{u}}^\varepsilon\otimes\overline{\mathbf{u}}^\varepsilon + \nabla\overline{\mathbf{d}}^\varepsilon\odot\nabla\overline{\mathbf{d}}^\varepsilon - \frac{1}{2}|\nabla\overline{\mathbf{d}}^\varepsilon|^2\mathbb{I}_2,\nabla\xi\rangle)dxds\Big] \\
&= \mathbb{E}'\Big[\int_D \langle \mathbf{u}(t)-\mathbf{u}_0,\xi\rangle dx \\
&\quad + \int_0^t\int_D (\langle \nabla\mathbf{u},\nabla\xi\rangle - \langle \mathbf{u}\otimes\mathbf{u}+\nabla\mathbf{d}\odot\nabla\mathbf{d}-\frac{1}{2}|\nabla\mathbf{d}|^2\mathbb{I}_2,\nabla\xi\rangle)dxds\Big] \\
&= \mathbb{E}'\left[\int_D \langle M_{\mathbf{u}}(t),\xi\rangle dx\right].
\end{aligned}
$$

Now we turn to the convergence of $\overline{\mathbf{d}}^\varepsilon$. We claim that up to a subsequence,

$$
\Delta\overline{\mathbf{d}}^\varepsilon - \mathbf{f}_\varepsilon(\overline{\mathbf{d}}^\varepsilon) \rightharpoonup \Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d} \text{ in } L^2(\Omega'\times[0,T]\times D). \tag{42}
$$

From (16) we can assume that there exists $\mathbf{g}\in L^2(\Omega'\times[0,T]\times D)$ such that

$$
\Delta\overline{\mathbf{d}}^\varepsilon - \mathbf{f}_\varepsilon(\overline{\mathbf{d}}^\varepsilon) \rightharpoonup \mathbf{g} \text{ in } L^2(\Omega'\times[0,T]\times D).
$$

First we claim that

$$
\mathbf{g}\perp\mathbf{d} \text{ for almost all } (\omega',t,x)\in\Omega'\times[0,T]\times D. \tag{43}
$$

In fact, for any test function $\phi=\phi(\omega',x)$, if we apply the Itô formula to

$$
\Psi(\overline{\mathbf{d}}^\varepsilon) = \int_D \frac{|\overline{\mathbf{d}}^\varepsilon|^2}{2}\phi dx,
$$

it hold that (see Appendix A)

$$
\begin{aligned}
&\mathbb{E}'\left[\int_D \frac{|\overline{\mathbf{d}}^\varepsilon|^2(t)}{2}\phi dx\right] - \mathbb{E}'\left[\int_D \frac{|\overline{\mathbf{d}}^\varepsilon|^2(t-\delta)}{2}\phi dx\right] \\
&= -\mathbb{E}'\left[\int_{t-\delta}^t\int_D \phi\overline{\mathbf{u}}^\varepsilon\cdot\nabla\frac{|\overline{\mathbf{d}}^\varepsilon|^2}{2}dxds\right] + \mathbb{E}'\left[\int_{t-\delta}^t\int_D \langle \Delta\overline{\mathbf{d}}^\varepsilon - \mathbf{f}_\varepsilon(\overline{\mathbf{d}}^\varepsilon),\overline{\mathbf{d}}^\varepsilon\rangle\phi dxds\right].
\end{aligned}
$$

Now we pass ε to 0, using the fact that $|\mathbf{d}|=1$ for almost all $(\omega',t,x)\in\Omega'\times[0,T]\times D$ we get

$$
\mathbb{E}'\left[\int_{t-\delta}^t\int_D \langle \mathbf{g},\mathbf{d}\rangle\phi dxds\right] = 0. \tag{44}
$$

Since ϕ and δ can be arbitrary, $\langle\mathbf{g},\mathbf{d}\rangle=0$ for almost all $(\omega',t,x)\in\Omega'\times[0,T]\times D$. Hence (43) holds. By taking the cross product of (42) with $\overline{\mathbf{d}}^\varepsilon\phi$ we get

$$
\begin{aligned}
0 &= \lim_{\varepsilon\to 0}\mathbb{E}'\left[\int_0^T\int_D \langle(\Delta\overline{\mathbf{d}}^\varepsilon - \mathbf{f}_\varepsilon(\overline{\mathbf{d}}^\varepsilon)-\mathbf{g})\times\overline{\mathbf{d}}^\varepsilon,\phi\rangle dxdt\right] \\
&= \lim_{\varepsilon\to 0}\mathbb{E}'\left[\int_0^T\int_D \langle\nabla\cdot(\nabla\overline{\mathbf{d}}^\varepsilon\times\overline{\mathbf{d}}^\varepsilon),\phi\rangle dxdt - \int_0^T\int_D\langle\mathbf{g}\times\overline{\mathbf{d}}^\varepsilon,\phi\rangle dxdt\right] \\
&= \lim_{\varepsilon\to 0}\mathbb{E}'\left[-\int_0^T\int_D \langle\nabla\overline{\mathbf{d}}^\varepsilon\times\overline{\mathbf{d}}^\varepsilon,\nabla\phi\rangle dxdt - \int_0^T\int_D\langle\mathbf{g}\times\overline{\mathbf{d}}^\varepsilon,\phi\rangle dxdt\right] \\
&= \mathbb{E}'\left[-\int_0^T\int_D \langle\nabla\mathbf{d}\times\mathbf{d},\nabla\phi\rangle dxdt - \int_0^T\int_D\langle\mathbf{g}\times\mathbf{d},\phi\rangle dxdt\right].
\end{aligned}
$$

This implies $(\mathbf{g}-\Delta\mathbf{d})\times\mathbf{d}=0$ and hence there exists $\lambda=\lambda(\omega',t,x):\Omega'\times[0,T]\times D\to\mathbb{R}$ such that

$$\mathbf{g}-\Delta\mathbf{d}=\lambda\mathbf{d}.$$

From (43) and $\langle\Delta\mathbf{d},\mathbf{d}\rangle=-|\nabla\mathbf{d}|^2\mathbf{d}$ we get

$$\lambda=\langle\mathbf{g}-\Delta\mathbf{d},\mathbf{d}\rangle=|\nabla\mathbf{d}|^2.$$

Thus (42) holds. By (22), (40) and (42), we have for any $\zeta\in L^2(\Omega';H^1(D,\mathbb{R}^3))$, it holds

$$\begin{aligned}
&\lim_{\varepsilon\to0}\mathbb{E}'\left[\int_D\langle M_{\overline{\mathbf{d}}^\varepsilon}(t),\zeta\rangle dx\right] \qquad (45)\\
&=\lim_{\varepsilon\to0}\mathbb{E}'\Big[\int_D\langle\overline{\mathbf{d}}^\varepsilon(t)-\mathbf{d}_0^\varepsilon,\zeta\rangle dx\\
&-\int_0^t\int_D(\langle\overline{\mathbf{u}}^\varepsilon\otimes\overline{\mathbf{d}}^\varepsilon,\nabla\zeta\rangle-\langle\Delta\overline{\mathbf{d}}^\varepsilon-\mathbf{f}_\varepsilon(\overline{\mathbf{d}}^\varepsilon),\zeta\rangle)dxds\Big]\\
&-\lim_{\varepsilon\to0}\mathbb{E}'\left[\frac12\int_0^t\int_D\langle(\overline{\mathbf{d}}^\varepsilon\times\mathbf{h})\times\mathbf{h},\zeta\rangle dxds\right]\\
&=\mathbb{E}'\left[\int_D\langle\mathbf{d}(t)-\mathbf{d}_0,\zeta\rangle dx-\int_0^t\int_D(\langle\mathbf{u}\otimes\mathbf{d},\nabla\zeta\rangle-\langle\Delta\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d},\zeta\rangle)dxds\right]\\
&\quad-\mathbb{E}'\left[\frac12\int_0^t\int_D\langle(\mathbf{d}\times\mathbf{h})\times\mathbf{h},\zeta\rangle dxds\right]\\
&=\mathbb{E}'\left[\int_D\langle M_{\mathbf{d}}(t),\zeta\rangle dx\right].
\end{aligned}$$

Taking the limit $\varepsilon\to0$ in (30) and applying the lower semicontinuity yields (5).

To finish the construction, we need to show that for every $t\in(0,T]$

$$\int_0^t S(\overline{\mathbf{u}}^\varepsilon)d\overline{W}_1^\varepsilon(s)\to\int_0^t S(\mathbf{u})dW_1'(s)\text{ in }L^2(\Omega;L^2(D)), \qquad (46)$$

$$\int_0^t(\overline{\mathbf{d}}^\varepsilon\times\mathbf{h})d\overline{W}_2^\varepsilon(s)\to\int_0^t(\mathbf{d}\times\mathbf{h})dW_2'(s)\text{ in }L^2(\Omega;L^2(D)). \qquad (47)$$

For this purpose, we adapt the strategy from [6]. Let $\mathcal{N}$ be the set of null sets of $\mathcal{F}'$ and for any $t\ge0$ and $\varepsilon>0$, let

$$\begin{aligned}
\hat{\mathcal{F}}_t^\varepsilon&:=\sigma\left(\sigma\left((\overline{\mathbf{u}}^\varepsilon(s),\overline{\mathbf{d}}^\varepsilon(s),\overline{W}_1^\varepsilon(s),\overline{W}_2^\varepsilon(s));s\le t\right)\cup\mathcal{N}\right),\\
\mathcal{F}_t'&:=\sigma\left(\sigma\left((\mathbf{u}(s),\mathbf{d}(s),W_1'(s),W_2'(s));s\le t\right)\cup\mathcal{N}\right).
\end{aligned}$$

Since $\mathcal{L}(\overline{\mathbf{u}}^\varepsilon,\overline{\mathbf{d}}^\varepsilon,\overline{W}_1^\varepsilon,\overline{W}_2^\varepsilon)=\mathcal{L}(\mathbf{u}^\varepsilon,\mathbf{d}^\varepsilon,W_1,W_2)$, $(\overline{W}_1^\varepsilon,\overline{W}_2^\varepsilon)$ form a sequence of cylindrical Wiener processes. Moreover, for $0\le s<t\le T$ the increments $(\overline{W}_1^\varepsilon(t)-\overline{W}_1^\varepsilon(s),\overline{W}_2^\varepsilon(t)-\overline{W}_2^\varepsilon(s))$ are independent of $\hat{\mathcal{F}}_r^\varepsilon$ for $r\in[0,s]$. Let $k\in\mathbb{N}$ and $s_0=0<s_1<\cdots<s_k\le T$ be a partition of $[0,T]$. By the characterization of K_2-valued K_1-cylindrical Wiener process [6, Remark 2.8], for each $\xi\in K_2^*$ we have

$$\begin{aligned}
\mathbb{E}'\left[e^{i\sum_{j=1}^k\left\langle\xi,\overline{W}_1^\varepsilon(s_j)-\overline{W}_1^\varepsilon(s_{j-1})\right\rangle_{K_2^*,K_2}}\right]&=\mathbb{E}\left[e^{i\sum_{j=1}^k\langle\xi,W_1(s_j)-W_1(s_{j-1})\rangle_{K_2^*,K_2}}\right]\\
&=e^{-\frac12\sum_{j=1}^k(s_j-s_{j-1})|\xi|_{K_1}^2}.
\end{aligned}$$

Thanks to (22) and the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{\varepsilon\to 0}\mathbb{E}'\left[e^{i\sum_{j=1}^{k}\left\langle\xi,\overline{W}_1^{\varepsilon}(s_j)-\overline{W}_1^{\varepsilon}(s_{j-1})\right\rangle_{K_2^*,K_2}}\right]=\mathbb{E}'\left[e^{i\sum_{j=1}^{k}\left\langle\xi,W_1'(s_1)-W_1'(s_{j-1})\right\rangle_{K_2^*,K_2}}\right]$$
$$=e^{-\frac{1}{2}\sum_{j=1}^{k}(s_j-s_{j-1})|\xi|_{K_1}^2}.$$

Hence the finite dimensional distribution of W_1' is Gaussian. The same argument also works for W_2'. Next we want to show that $(W_1'(t)-W_1'(s),W_2'(t)-W_2'(s)),0\le s<t\le T$ is independent of $\mathcal{F}_r'$ for $r\in[0,s]$. Consider $\{\phi_j\}_{j=1}^{k}\in C_b(W^{-2,\tilde{p}}(D)\times L^{\frac{4}{3}}(D)),\{\psi_j\}_{j=1}^{k}\in C_b(K_2\times\mathbb{R})$, let $0\le t_1<\cdots<r_k\le s<t\le T$, $\psi\in C_b(K_2),\zeta\in C_b(\mathbb{R})$.

$$\mathbb{E}'\Bigg[\left(\prod_{j=1}^{k}\phi_j(\overline{\mathbf{u}}^{\varepsilon}(r_j),\overline{\mathbf{d}}^{\varepsilon}(r_j))\prod_{j=1}^{k}\psi_j(\overline{W}_1^{\varepsilon}(r_j),\overline{W}_2^{\varepsilon}(r_j))\right)$$
$$\times\psi(\overline{W}_1^{\varepsilon}(t)-\overline{W}_1^{\varepsilon}(s))\zeta(\overline{W}_2^{\varepsilon}(t)-\overline{W}_2^{\varepsilon}(s))\Bigg] \tag{48}$$
$$=\mathbb{E}'\left[\prod_{j=1}^{k}\phi_j(\overline{\mathbf{d}}^{\varepsilon}(r_j),\overline{\mathbf{d}}^{\varepsilon}(r_j))\prod_{j=1}^{k}\psi_j(\overline{W}_1^{\varepsilon}(r_j),\overline{W}_2^{\varepsilon}(r_j))\right]$$
$$\times\mathbb{E}'\left[\psi(\overline{W}_1^{\varepsilon}(t)-\overline{W}_1^{\varepsilon}(s))\right]\mathbb{E}'\left[\zeta(\overline{W}_2^{\varepsilon}(t)-\overline{W}_2^{\varepsilon}(s))\right].$$

Again by the Lebesgue Dominated Convergence theorem, if we send $\varepsilon\to 0$ in (48) we can see (48) also holds for $(\mathbf{u},\mathbf{d},W_1',W_2')$ in the limit. Furthermore, it is easy to show that W_1' is independent of W_2'.

For any $\delta>0$, let η_δ be a standard mollifier with support in $(0,t)$. Define

$$S^{\delta}(\mathbf{u}(s))=\int_{-\infty}^{\infty}\eta_\delta(s-r)S(\mathbf{u}(r)dr.$$

Let $M_{\overline{\mathbf{u}}^{\varepsilon}}^{\delta}$ and $M_{\mathbf{u}}^{\delta}$ be respectively defined by

$$M_{\overline{\mathbf{u}}^{\varepsilon}}^{\delta}(t)=\int_0^t S^{\delta}(\overline{\mathbf{u}}^{\varepsilon}(s))d\overline{W}_1^{\varepsilon}(s),$$
$$M_{\mathbf{u}}^{\delta}(t)=\int_0^t S^{\delta}(\mathbf{u}(s))dW_1'(s).$$

By the property of mollifiers, we can get for any $\mathbf{v}\in\mathbf{H}$

$$\lim_{\delta\to 0}\mathbb{E}'\int_0^t\|S^{\delta}(\mathbf{v}(s))-S(\mathbf{v}(s))\|_{\mathcal{L}_2(K_1,\mathbf{H})}^2ds=0.$$

Hence, for any $t\in(0,T]$, we have the following uniform approximation

$$\lim_{\delta\to 0}\sup_{0<\varepsilon<1}\mathbb{E}'\left\|M_{\overline{\mathbf{u}}^{\varepsilon}}^{\delta}(t)-\int_0^t S(\overline{\mathbf{u}}^{\varepsilon})d\overline{W}_1^{\varepsilon}(s)\right\|^2=0, \tag{49}$$

and

$$\lim_{\delta\to 0}\mathbb{E}'\left\|M_{\mathbf{u}}^{\delta}(t)-\int_0^t S(\mathbf{u})dW_1'(s)\right\|^2=0. \tag{50}$$

Next, we need to show that for any $\delta>0$

$$\lim_{\varepsilon\to 0}\mathbb{E}'\left\|M_{\overline{\mathbf{u}}^{\varepsilon}}^{\delta}(t)-M_{\mathbf{u}}^{\delta}(t)\right\|^2=0. \tag{51}$$

If we write $\overline{W}_1^{\varepsilon}(t) = \sum_{i=1}^{\infty} \overline{B}_i^{\varepsilon}(t)e_i$ and $W_1'(t) = \sum_{i=1}^{\infty} B_i'(t)e_i$, where $\{\overline{B}_i^{\varepsilon}\}_{i=1}^{\infty}$, $\{B_i'(t)\}_{i=1}^{\infty}$ are i.i.d. stardard Brownian motions, then

$$M^{\delta}_{\overline{\mathbf{u}}^{\varepsilon}}(t) - M^{\delta}_{\mathbf{u}}(t) = \sum_{i=1}^{\infty}\int_0^t S^{\delta}(\overline{\mathbf{u}}^{\varepsilon}(s))(e_i)d\overline{B}_i^{\varepsilon}(s) - \sum_{i=1}^{\infty}\int_0^t S^{\delta}(\mathbf{u}(s))(e_i)dB_i'(s).$$

By Young's convolution inequality, we have that

$$\mathbb{E}'\int_0^t \|S^{\delta}(\mathbf{u}(s))\|^2_{\mathcal{L}_2(K_1,\mathbf{H})}ds \leq C\mathbb{E}'\int_0^t \|S(\mathbf{u}(s))\|^2_{\mathcal{L}_2(K_1,\mathbf{H})}ds \leq C.$$

Thus, for any $\gamma > 0$, there exists an $N \in \mathbb{N}_+$ such that

$$\sum_{i=N+1}^{\infty}\mathbb{E}'\int_0^t \|S^{\delta}(\mathbf{u}(s))(e_i)\|^2 ds < \gamma.$$

Since

$$\lim_{\varepsilon\to 0}\mathbb{E}'\int_0^t \|S^{\delta}(\overline{\mathbf{u}}^{\varepsilon}(s)) - S^{\delta}(\mathbf{u}(s))\|^2_{\mathcal{L}_2(K_1,\mathbf{H})}ds = 0,$$

there exists an $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$,

$$\sum_{i=N+1}^{\infty}\mathbb{E}'\int_0^t \|S^{\delta}(\overline{\mathbf{u}}^{\varepsilon}(s))(e_i)\|^2 ds < 2\gamma.$$

Now we split $M^{\delta}_{\overline{\mathbf{u}}^{\varepsilon}}(t) - M^{\delta}_{\mathbf{u}}(t)$ into three parts

$$\begin{aligned}
M^{\delta}_{\overline{\mathbf{u}}^{\varepsilon}}(t) - M^{\delta}_{\mathbf{u}}(t) &= \sum_{i=1}^{N}\left(\int_0^t S^{\delta}(\overline{\mathbf{u}}^{\varepsilon}(s))(e_i)d\overline{B}_i^{\varepsilon}(s) - \int_0^t S^{\delta}(\mathbf{u}(s))(e_i)dB_i'(s)\right)\\
&+ \sum_{i=N+1}^{\infty}\int_0^t S^{\delta}(\overline{\mathbf{u}}^{\varepsilon}(s))(e_i)d\overline{B}_i^{\varepsilon}(s)\\
&+ \sum_{i=N+1}^{\infty}\int_0^t S^{\delta}(\mathbf{u}(s))(e_i)dB_i'(s) := J^{\delta}_{\varepsilon,1}(t) + J^{\delta}_{\varepsilon,2}(t) + J^{\delta}_{\varepsilon,3}(t).
\end{aligned}$$

By the Itô isometry, we have that

$$\begin{aligned}
\mathbb{E}'\|J^{\delta}_{\varepsilon,2}\|^2 &= \sum_{i=N+1}^{\infty}\mathbb{E}'\int_0^t \|S^{\delta}(\overline{\mathbf{u}}^{\varepsilon}(s))(e_i)\|^2 ds < 2\gamma,\\
\mathbb{E}'\|J^{\delta}_{\varepsilon,3}\|^2 &= \sum_{i=N+1}^{\infty}\mathbb{E}'\int_0^t \|S^{\delta}(\mathbf{u}(s))(e_i)\|^2 ds < \gamma.
\end{aligned}$$

For $J_{\varepsilon,1}(t)$, we write

$$\begin{aligned}
J^{\delta}_{\varepsilon,1}(t) &= \sum_{i=1}^{N}\left(\int_0^t S^{\delta}(\overline{\mathbf{u}}^{\varepsilon}(s))(e_i)d\overline{B}_i^{\varepsilon}(s) - \int_0^t S^{\delta}(\overline{\mathbf{u}}^{\varepsilon}(s))(e_i)dB_i'(s)\right)\\
&+ \sum_{i=1}^{N}\left(\int_0^t S^{\delta}(\overline{\mathbf{u}}^{\varepsilon}(s))(e_i)dB_i'(s) - \int_0^t S^{\delta}(\mathbf{u}(s))(e_i)dB_i'(s)\right)\\
&:= I^{\delta}_{\varepsilon,1} + I^{\delta}_{\varepsilon,2}.
\end{aligned}$$

For $I^{\delta}_{\varepsilon,1}(t)$, by integration by parts we obtain that

$$I^{\delta}_{\varepsilon,1}(t) = \sum_{i=1}^{N}\left(\int_0^t [\eta'_\delta \star S(\overline{\mathbf{u}}^\varepsilon(s))](e_i)B'_i(s)ds - \int_0^t [\eta'_\delta \star S(\overline{\mathbf{u}}^\varepsilon(s))](e_i)\overline{B}^\varepsilon_i(s)ds\right)$$
$$= -\sum_{i=1}^{N}\left(\int_0^t [\eta'_\delta \star S(\overline{\mathbf{u}}^\varepsilon(s))](e_i)[\overline{B}^\varepsilon_i(s) - B'_i(s)]ds\right).$$

From the Burkholder–Davis–Gundy inequality, we get for any $p > 1$, any $i = 1, 2, ..., N$,

$$\sup_{\varepsilon>0}\mathbb{E}'\sup_{s\in[0,T]}\left(|\overline{B}^\varepsilon_i(s)|^p + |B'_i(s)|^p\right) \le CT^{\frac{p}{2}}. \tag{52}$$

Hence, by the uniform integrability (52) and the almost surely convergence (22) we have that for $i = 1, 2, \dots, N$,

$$\lim_{\varepsilon\to 0}\mathbb{E}'\int_0^t |\overline{B}^\varepsilon_i(s) - B'_i(s)|^p ds = 0.$$

This implies

$$\mathbb{E}'\|I^{\delta}_{\varepsilon,1}(t)\|^2 = \mathbb{E}'\left\|\sum_{i=1}^{N}\int_0^t [\eta'_\delta \star S(\overline{\mathbf{u}}^\varepsilon(s))](e_i)(\overline{B}^\varepsilon_i(s) - B'_i(s))ds\right\|^2$$
$$\le N\sum_{i=1}^{N}\mathbb{E}'\left\|\int_0^t [\eta'_\delta \star S(\overline{\mathbf{u}}^\varepsilon(s))](e_i)(\overline{B}^\varepsilon_i(s) - B'_i(s))ds\right\|^2$$
$$\le N\sum_{i=1}^{N}\mathbb{E}'\left[\int_0^t \|[\eta'_\delta \star S(\overline{\mathbf{u}}^\varepsilon(s))](e_i)\||\overline{B}^\varepsilon_i(s) - B'_i(s)|ds\right]^2$$
$$\le N\sum_{i=1}^{N}\mathbb{E}'\left[\int_0^t \|\eta'_\delta \star S(\overline{\mathbf{u}}^\varepsilon(s))(e_i)\|^2 ds\int_0^t |\overline{B}^\varepsilon_i(s) - B'_i(s)|^2 ds\right]$$
$$\le \frac{CN}{\delta^2}\sum_{i=1}^{N}\mathbb{E}'\left[\int_0^t \|S(\overline{\mathbf{u}}^\varepsilon)(s)\|^2_{\mathcal{L}_2(K_1,\mathbf{H})}ds\int_0^t |\overline{B}^\varepsilon_i(s) - B'_1(s)|^2 ds\right]$$
$$\le \frac{CN}{\delta^2}\sum_{i=1}^{N}\mathbb{E}'\left[\int_0^t (1 + \|\overline{\mathbf{u}}^\varepsilon\|^2)ds\int_0^t |\overline{B}^\varepsilon_i(s) - B'_i(s)|^2 ds\right]$$
$$\le \frac{CNT^{\frac{3}{2}}}{\delta^2}\left(\mathbb{E}'\sup_{0\le s\le t}(1 + \|\overline{\mathbf{u}}^\varepsilon(s)\|)^4\right)^{\frac{1}{2}}\left(\mathbb{E}'\int_0^t |\overline{B}^\varepsilon_i(s) - B'_i(s)|^4 ds\right)^{\frac{1}{2}}$$
$$\le \frac{CNT^{\frac{3}{2}}}{\delta^2}\sum_{i=1}^{N}\left(\mathbb{E}'\int_0^t |\overline{B}^\varepsilon_i(s) - B'_i(s)|^4 ds\right)^{\frac{1}{2}} \to 0,$$

as $\varepsilon \to 0$. Using a similar argument, we can show that

$$\lim_{\varepsilon\to 0}\mathbb{E}'\|I^{\delta}_{\varepsilon,2}(t)\|^2 = 0.$$

Since γ can be arbitrarily small, we get

$$\lim_{\varepsilon\to 0}\mathbb{E}'\left[\|J^{\delta}_{\varepsilon,1}\|^2 + \|J^{\delta}_{\varepsilon,2}(t)\|^2 + \|J^{\delta}_{\varepsilon,3}(t)\|^2\right] = 0,\ \forall t \in (0, T],$$

This implies (51). Then we can conclude from (49), (50) and (51) that for every $t \in (0,T]$,

$$\lim_{\varepsilon\to 0}\mathbb{E}'\left\|\int_0^t S(\overline{\mathbf{u}}^\varepsilon(s))d\overline{W}_1^\varepsilon(s)-\int_0^t S(\mathbf{u}(s))dW_1'(s)\right\|^2=0.$$

Similarly, we can show

$$\lim_{\varepsilon\to 0}\mathbb{E}'\left\|\int_0^t (\overline{\mathbf{d}}^\varepsilon\times\mathbf{h})d\overline{W}_2^\varepsilon(s)-\int_0^t (\mathbf{d}\times\mathbf{h})dW_2'(s)\right\|^2=0.$$

Hence, the convergence of martingale terms (46) and (47) holds. Putting (41), (45), (46) and (47) together completes the proof.

Appendix A. **Itô's formulas for functionals of d.** Consider the functional

$$\Psi(\mathbf{d}^\varepsilon):=\int_D \frac{|\mathbf{d}^\varepsilon|^2}{2}\phi dx.$$

It is easy to obtain the first and and second Fréchet derivatives of $\Psi(\mathbf{d}^\varepsilon)$

$$\Psi'(\mathbf{d}^\varepsilon)[\mathbf{g}]=\int_D\langle\mathbf{d}^\varepsilon,\mathbf{g}\rangle\phi dx,$$
$$\Psi''(\mathbf{d}^\varepsilon)[\mathbf{g},\mathbf{g}]=\int_D\langle\mathbf{g},\mathbf{g}\rangle\phi dx.$$

Applying the Itô formula to $\Psi(\mathbf{d}^\varepsilon)$ gives

$$d\Psi(\mathbf{d}^\varepsilon)=\Psi'(\mathbf{d}^\varepsilon)[d\mathbf{d}^\varepsilon]+\frac{1}{2}\Psi_\varepsilon''(\mathbf{d}^\varepsilon)[d\mathbf{d}^\varepsilon,d\mathbf{d}^\varepsilon].$$

Since,

$$d\mathbf{d}^\varepsilon=\underbrace{\left(-\mathbf{u}^\varepsilon\cdot\nabla\mathbf{d}^\varepsilon+\Delta\mathbf{d}^\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)+\frac{1}{2}(\mathbf{d}^\varepsilon\times\mathbf{h})\times\mathbf{h}\right)}_{\mathbf{j}}dt+\underbrace{(\mathbf{d}^\varepsilon\times\mathbf{h})}_{\mathbf{k}}dW_2,$$

we then obtain that for $0<\delta<t$,

$$\begin{aligned}
&\Psi(\mathbf{d}^\varepsilon)(t)-\Psi(\mathbf{d}^\varepsilon)(t-\delta)=\int_{t-\delta}^t\left(\Psi'(\mathbf{d}^\varepsilon)[\mathbf{j}(s)]+\frac{1}{2}\Psi''(\mathbf{d}^\varepsilon)[\mathbf{k}(s),\mathbf{k}(s)]\right)ds\\
&+\int_{t-\delta}^t\Psi'(\mathbf{d}^\varepsilon)[\mathbf{k}(s)]dW_2(s)\\
&=\int_{t-\delta}^t\int_D\langle-\mathbf{u}^\varepsilon\cdot\nabla\mathbf{d}^\varepsilon,\mathbf{d}^\varepsilon\rangle\phi dxds+\int_{t-\delta}^t\int_D\langle\Delta\mathbf{d}^\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon),\mathbf{d}^\varepsilon\rangle\phi dxds\\
&+\frac{1}{2}\int_{t-\delta}^t\int_D\langle(\mathbf{d}^\varepsilon\times\mathbf{h})\times\mathbf{h},\mathbf{d}^\varepsilon\rangle\phi dxds+\frac{1}{2}\int_{t-\delta}^t\int_D|\mathbf{d}^\varepsilon\times\mathbf{h}|^2\phi dxds\\
&+\int_{t-\delta}^t\int_D\langle\mathbf{d}^\varepsilon\times\mathbf{h},\mathbf{d}^\varepsilon\rangle\phi dxdW_2(s)\\
&=\int_{t-\delta}^t-\mathbf{u}^\varepsilon\cdot\nabla\frac{|\mathbf{d}^\varepsilon|^2}{2}\phi dxds+\int_{t-\delta}^t\int_D\langle\Delta\mathbf{d}^\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon),\mathbf{d}^\varepsilon\rangle\phi dxds,
\end{aligned}$$

where we use the fact the vector triple product

$$\langle(\mathbf{d}^\varepsilon\times\mathbf{h})\times\mathbf{h},\mathbf{d}^\varepsilon\rangle=-|\mathbf{d}^\varepsilon\times\mathbf{h}|^2. \tag{53}$$

and

$$\langle \mathbf{d}^\varepsilon \times \mathbf{h}, \mathbf{d}^\varepsilon \rangle = 0. \tag{54}$$

Recall the energy functional

$$\Phi_\varepsilon(\mathbf{d}^\varepsilon) = \frac{1}{2}\|\nabla \mathbf{d}^\varepsilon\|^2 + \int_D F_\varepsilon(\mathbf{d}^\varepsilon)dx.$$

The first and second Fréchet derivatives of Φ_ε are given by

$$\begin{aligned}
\Phi'_\varepsilon(\mathbf{d}^\varepsilon)[\mathbf{g}] &= \int_D \left(\langle \nabla \mathbf{d}^\varepsilon, \nabla \mathbf{g}\rangle + \langle \mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon), \mathbf{g}\rangle\right) dx \\
&= \int_D \langle -\Delta \mathbf{d}^\varepsilon + \frac{|\mathbf{d}^\varepsilon|^2-1}{\varepsilon^2}\mathbf{d}^\varepsilon, \mathbf{g}\rangle dx, \\
\Phi''_\varepsilon(\mathbf{d}^\varepsilon)[\mathbf{g},\mathbf{g}] &= \int_D \left(\langle \nabla \mathbf{g}, \nabla \mathbf{g}\rangle + \frac{|\mathbf{d}^\varepsilon|^2-1}{\varepsilon^2}|\mathbf{g}|^2 + \frac{2}{\varepsilon^2}\langle \mathbf{d}^\varepsilon, \mathbf{g}\rangle^2\right) dx
\end{aligned}$$

for every $\mathbf{g} \in H^1(D;\mathbb{R}^3)$. Then, the Itô formula for $\Phi_\varepsilon(\mathbf{d}^\varepsilon)$ reads

$$d\Phi_\varepsilon(\mathbf{d}^\varepsilon) = \Phi'_\varepsilon(\mathbf{d}^\varepsilon)[d\mathbf{d}^\varepsilon] + \frac{1}{2}\Phi''_\varepsilon(\mathbf{d}^\varepsilon)[d\mathbf{d}^\varepsilon, d\mathbf{d}^\varepsilon].$$

From the identity (53) and (54) we obtain

$$\begin{aligned}
&\Phi_\varepsilon(\mathbf{d}^\varepsilon)(t) - \Phi_\varepsilon(\mathbf{d}_0) \\
&= \int_0^t \left(\Phi'_\varepsilon(\mathbf{d}^\varepsilon)[\mathbf{j}(s)] + \frac{1}{2}\Phi''_\varepsilon(\mathbf{d}^\varepsilon)[\mathbf{k}(s),\mathbf{k}(s)]\right) ds + \int_0^t \Phi'_\varepsilon(\mathbf{d}^\varepsilon)[\mathbf{k}(s)]dW_2(s) \\
&= \int_0^t \int_D \langle \mathbf{u}^\varepsilon \cdot \nabla \mathbf{d}^\varepsilon, \Delta \mathbf{d}^\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)\rangle dxds - \int_0^t \int_D |\Delta \mathbf{d}^\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)|^2 dxds \\
&+ \frac{1}{2}\int_0^t \int_D \langle -\Delta \mathbf{d}^\varepsilon + \frac{|\mathbf{d}^\varepsilon|^2-1}{\varepsilon^2}\mathbf{d}^\varepsilon, (\mathbf{d}^\varepsilon \times \mathbf{h}) \times \mathbf{h}\rangle dxds \\
&+ \frac{1}{2}\int_0^t \int_D \left(|\nabla(\mathbf{d}^\varepsilon \times \mathbf{h})|^2 + \frac{|\mathbf{d}^\varepsilon|^2-1}{\varepsilon^2}|\mathbf{d}^\varepsilon \times \mathbf{h}|^2\right) dxds \\
&+ \frac{1}{2}\int_0^t \int_D \langle -\Delta \mathbf{d}^\varepsilon + \mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon), \mathbf{d}^\varepsilon \times \mathbf{h}\rangle dxdW_2(s) \\
&= \int_0^t \int_D \langle \mathbf{u}^\varepsilon \cdot \nabla \mathbf{d}^\varepsilon, \Delta \mathbf{d}^\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)\rangle dxds - \int_0^t \int_D |\Delta \mathbf{d}^\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)|^2 dxds \\
&+ \frac{1}{2}\int_0^t \int_D (\langle \nabla \mathbf{d}^\varepsilon, \nabla((\mathbf{d}^\varepsilon \times \mathbf{h}) \times \mathbf{h})\rangle + |\nabla(\mathbf{d}^\varepsilon \times \mathbf{h})|^2) dxds \\
&+ \frac{1}{2}\int_0^t \int_D \langle -\Delta \mathbf{d}^\varepsilon + \mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon), \mathbf{d}^\varepsilon \times \mathbf{h}\rangle dxdW_2(s).
\end{aligned}$$

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