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Partial regularity of a nematic liquid crystal model with kinematic transport effects

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Partial regularity of a nematic liquid crystal model with kinematic transport effects

Hengrong Du[*] and Changyou Wang

Department of Mathematics, Purdue University, West Lafayette, IN 47906, United States of America

E-mail: du155@purdue.edu

Abstract

In this paper, we will establish the global existence of a suitable weak solution to the Erickson–Leslie system modelling hydrodynamics of nematic liquid crystal flows with kinematic transports for molecules of various shapes in $\mathbb{R}^3$, which is smooth away from a closed set of (parabolic) Hausdorff dimension at most $\frac{15}{7}$.

1. Introduction

In this paper, we will study the simplified Ericksen–Leslie system modelling the hydrodynamics of nematic liquid crystals with variable degrees of orientation and kinematic transports for molecules of various shapes: $(\mathbf{u}, \mathbf{d}, P) : \mathbb{R}^3 \times (0, \infty) \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$ solves

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \nu \Delta \mathbf{u} - \lambda \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d} + S_\alpha [\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}), \mathbf{d}]), \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} - T_\alpha [\nabla \mathbf{u}, \mathbf{d}] = \gamma(\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})), \end{cases} \tag{1.1}$$

where $\mathbf{u}(x, t)$ represents the velocity field of the flow, $\mathbf{d}(x, t)$ is the macroscopic averaged orientation field of the nematic liquid crystal modules, and P stands for the pressure function. Here $\mathbf{f}(\mathbf{d}) = D_{\mathbf{d}} F(\mathbf{d}) = (|\mathbf{d}|^2 - 1)\mathbf{d}$ is the gradient of Ginzburg–Landau potential function $F(\mathbf{d}) = \frac{1}{4}(1 - |\mathbf{d}|^2)^2$. Furthermore,

$$S_\alpha[\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}), \mathbf{d}] := \alpha(\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})) \otimes \mathbf{d} - (1 - \alpha)\mathbf{d} \otimes (\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})),$$

$$T_\alpha[\nabla \mathbf{u}, \mathbf{d}] := \alpha(\nabla \mathbf{u})\mathbf{d} - (1 - \alpha)(\nabla \mathbf{u})^T \mathbf{d},$$

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[*]Author to whom any correspondence should be addressed.

represents the Leslie stress tensor and the kinematic transport term respectively. The parameter $\alpha \in [0, 1]$ is the shape parameter of the liquid crystal molecule. In particular, $\alpha = 0, \frac{1}{2}$, and 1 corresponds to disc-like, spherical and rod-like molecule shape respectively (cf [5, 7, 9, 13]). The coefficient ν represents the fluid viscosity, λ stands for the competition between kinetic energy and potential energy, and γ reflects the molecular relaxation time.

In the 1960s, Ericksen and Leslie proposed a comprehensive hydrodynamic theory of nematic liquid crystals (cf [8, 17]). Since then there has been a great deal of theoretical and experimental work devoted to the study of nematic liquid crystal flows. The first rigorous mathematical study for the simplified Ericksen–Leslie system, that is, (1.1) without S_α, T_α terms, was made by Lin–Liu [19]. Later in [20], they established a partial regularity for the suitable weak solutions which satisfy the local energy inequality, analogous to the Navier–Stokes equations by Caffarelli–Kohn–Nirenberg in [3]. Very recently, the same type of regularity result was obtained for the co-rotational Beris–Edwards Q-tensor model by Du–Hu–Wang [6].

In this paper, we will construct a global-in-time suitable weak solution to (1.1), which enjoys a partial regularity that is slightly weaker than that of [20]. Besides its own interest, we believe that this partial regularity may be helpful to investigate the un-corotational Beris–Edwards system due to a similar structure of nonlinearities. There are two major difficulties in the analysis of (1.1):

- First, as pointed out by [24], when $\alpha \neq \frac{1}{2}$ the stretching effect induced by $T_\alpha[\nabla \mathbf{u}, \mathbf{d}]$ leads to the loss of maximum principle for the director field $\mathbf{d}$, which plays an essential role in [6, 20]. Here, inspired by [10, 18], we will prove an ε_0-regularity result by a blowing-up argument that involves a decay estimate of renormalized L^3-norm of both $|\nabla \mathbf{d}|$ and $|\mathbf{u}|$ and the mean oscillation of $\mathbf{d}$ in L^6 as well.
- Second, the presence of stress tensor $S_\alpha[\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}), \mathbf{d}]$ brings an extra difficulty on the decay estimate of renormalized $L^{\frac{3}{2}}$-norm of the pressure function P. While in the co-rotational regime, i.e., $\alpha = \frac{1}{2}$, we know that $S_{\frac{1}{2}}$ is anti-symmetric, which significantly simplifies the analysis on pressure function (see [6]).

We would like to mention that in a recent preprint [15], Koch obtained a partial regularity theorem for certain weak solutions to the Lin–Liu model that may be weaker than suitable weak solutions and may not obey the maximum principle, in which a smallness condition is imposed on normalized L^6-norm of $|\mathbf{d}|$.

Before stating our main results, we need to introduce

1.1. Some notations

For $\mathbf{u}, \mathbf{w} \in \mathbb{R}^3, A, B \in \mathbb{R}^{3 \times 3}$, we denote

$$\mathbf{u} \cdot \mathbf{w} := \sum_{i=1}^{3} \mathbf{u}_i \mathbf{w}_i, \qquad A : B := \sum_{i,j=1}^{3} A_{ij} B_{ij}, \qquad (A \cdot \mathbf{w})_j := \sum_{i=1}^{3} A_{ij} \mathbf{w}_i.$$

and

$$(\mathbf{u} \otimes \mathbf{w})_{ij} = \mathbf{u}_i \mathbf{w}_j, \qquad (\nabla \mathbf{d} \odot \nabla \mathbf{d})_{ij} = \sum_{k=1}^{3} \partial_i \mathbf{d}_k \partial_j \mathbf{d}_k,$$

$$[(\nabla \mathbf{u})\mathbf{d}]_i = \sum_{j=1}^{3} \partial_j \mathbf{u}_i \mathbf{d}_j, \qquad [(\nabla \mathbf{u})^T \mathbf{d}]_i = \sum_{i=1}^{3} \partial_i \mathbf{u}_j \mathbf{d}_j.$$

Define

$$\mathbf{H} = \text{Closure of } \left\{ \mathbf{u} \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^3) : \nabla \cdot \mathbf{u} = 0 \right\} \quad \text{in } L^2(\mathbb{R}^3),$$

and

$$\mathbf{V} = \text{Closure of } \left\{ \mathbf{u} \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^3) : \nabla \cdot \mathbf{u} = 0 \right\} \quad \text{in } H^1(\mathbb{R}^3).$$

For $0 \leqslant k \leqslant 5$, $\mathcal{P}^k$ denotes the k-dimensional Hausdorff measure on $\mathbb{R}^3 \times \mathbb{R}$ with respect to the parabolic distance:

$$\delta((x,t),(y,s)) = \max \left\{ |x - y|, \sqrt{|t - s|} \right\}, \ \forall \ (x,t),(y,s) \in \mathbb{R}^3 \times \mathbb{R}.$$

We let $B_r(x)$ denote the ball in $\mathbb{R}^3$ with centre x and radius r. For $z = (x,t) \in \mathbb{R}^3 \times \mathbb{R}_+$, denote $\mathbb{P}_r(z) := B_r(x) \times [t - r^2, t]$, and

$$f_{z,r} = \fint_{\mathbb{P}_r(z)} f := \frac{1}{|\mathbb{P}_r(z)|} \int_{\mathbb{P}_r(z)} f \, \mathrm{d}x \, \mathrm{d}t$$

for any function f on $\mathbb{P}_r(z)$.

Since the exact values of ν, λ, γ do not play roles in our analysis, we will assume

$$\nu = \lambda = \gamma = 1.$$

With the following identity

$$\nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}) = \nabla \mathbf{d} \cdot \Delta \mathbf{d} + \nabla \left(\frac{1}{2} |\nabla \mathbf{d}|^2 \right), \qquad \nabla F(\mathbf{d}) = \nabla \mathbf{d} \cdot \mathbf{f}(\mathbf{d}),$$

the system (1.1) can also be written as

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \Delta \mathbf{u} - \nabla \mathbf{d} \cdot (\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})) - \nabla \cdot S_\alpha[\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}), \mathbf{d}], \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} - T_\alpha[\nabla \mathbf{u}, \mathbf{d}] = \Delta \mathbf{d} - \mathbf{f}(\mathbf{d}). \end{cases}$$

$$(1.2)$$

subject to the initial condition

$$(\mathbf{u}, \mathbf{d}) \Big|_{t=0} = (\mathbf{u}_0, \mathbf{d}_0) \quad \text{in } \mathbb{R}^3. \tag{1.3}$$

Definition. A pair of functions $(\mathbf{u}, \mathbf{d}) : \mathbb{R}^3 \times (0, \infty) \to \mathbb{R}^3 \times \mathbb{R}^3$ is a weak solution of (1.2) and (1.3), if $(\mathbf{u}, \mathbf{d}) \in (L_t^\infty L_x^2 \cap L_t^2 H_x^1)(\mathbb{R}^3 \times (0, \infty), \mathbb{R}^3) \times (L_t^\infty H_x^1 \cap L_t^2 H_x^2)(\mathbb{R}^3 \times (0, \infty), \mathbb{R}^3)$, and for any $\phi \in C_0^\infty(\mathbb{R}^3 \times [0, \infty), \mathbb{R}^3)$ and $\psi \in C_0^\infty(\mathbb{R}^3 \times [0, \infty), \mathbb{R}^3)$, with $\mathrm{div}\,\phi = 0$ in $\mathbb{R}^3 \times [0, \infty)$, it holds that

$$\int_{\mathbb{R}^3 \times (0,\infty)} [-\mathbf{u} \cdot \partial_t \phi + \nabla \mathbf{u} : \nabla \phi - \mathbf{u} \otimes \mathbf{u} : \nabla \phi - (\phi \cdot \nabla \mathbf{d}) \cdot (\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}))] \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_{\mathbb{R}^3 \times (0,\infty)} S_\alpha[\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}), \mathbf{d}] : \nabla \phi \, \mathrm{d}x \, \mathrm{d}t = \int_{\mathbb{R}^3} \mathbf{u}_0 \cdot \phi(x, 0) \mathrm{d}x, \tag{1.4}$$

$$\int_{\mathbb{R}^3 \times (0,\infty)} [-\mathbf{d} \cdot \partial_t \psi + \nabla \mathbf{d} : \nabla \psi - \mathbf{d} \otimes \mathbf{u} : \nabla \psi + \mathbf{f}(\mathbf{d}) \cdot \psi] \, dx \, dt$$

$$- \int_{\mathbb{R}^3 \times (0,\infty)} T_\alpha [\nabla \mathbf{u}, \mathbf{d}] \cdot \psi dx \, dt = \int_{\mathbb{R}^3} \mathbf{d}_0 \cdot \psi(x, 0) dx. \tag{1.5}$$

The global and local energy inequalities for (1.2) play the basic roles: for $t > 0$,

$$\int_{\mathbb{R}^3} \left(\frac{1}{2} \left(|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2 \right) + F(\mathbf{d}) \right)(x, t) dx + \int_0^t \int_{\mathbb{R}^3} \left(|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})|^2 \right)(x, s) \, dx \, ds$$

$$\leqslant \int_{\mathbb{R}^3} \left(\frac{1}{2} (|\mathbf{u}_0|^2 + |\nabla \mathbf{d}_0|^2) + F(\mathbf{d}_0) \right)(x) dx, \tag{1.6}$$

$$\int_{\mathbb{R}^3} \left[\frac{1}{2} \left(|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2 \right) + F(\mathbf{d}) \right] \phi(x, t) dx + \int_0^t \int_{\mathbb{R}^3} \left(|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d}|^2 + |\mathbf{f}(\mathbf{d})|^2 \right) \phi(x, s) \, dx \, ds$$

$$\leqslant \int_0^t \int_{\mathbb{R}^3} \left[\frac{1}{2} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(\partial_t \phi + \Delta \phi) + F(\mathbf{d}) \partial_t \phi \right](x, s) \, dx \, ds$$

$$+ \int_0^t \int_{\mathbb{R}^3} \left[\frac{1}{2} \left(|\mathbf{u}|^2 + 2P \right) \mathbf{u} \cdot \nabla \phi + \nabla \mathbf{d} \odot \nabla \mathbf{d} : \mathbf{u} \otimes \nabla \phi \right](x, s) \, dx \, ds$$

$$+ \int_0^t \int_{\mathbb{R}^3} \left(\nabla \mathbf{d} \odot \nabla \mathbf{d} - |\nabla \mathbf{d}|^2 I_3 \right) : \nabla^2 \phi(x, s) dx ds$$

$$+ \int_0^t \int_{\mathbb{R}^3} S_\alpha [\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}), \mathbf{d}] : \mathbf{u} \otimes \nabla \phi(x, s) \, dx \, ds$$

$$+ \int_0^t \int_{\mathbb{R}^3} T_\alpha [\nabla \mathbf{u}, \mathbf{d}] \cdot (\nabla \phi \cdot \nabla \mathbf{d})(x, s) \, dx \, ds$$

$$- \int_0^t \int_{\mathbb{R}^3} (\nabla \phi \cdot \nabla \mathbf{d}) \cdot \mathbf{f}(\mathbf{d}) \, dx \, ds - 2 \int_0^t \int_{\mathbb{R}^3} \nabla \mathbf{f}(\mathbf{d}) : \nabla \mathbf{d} \phi(x, s) \, dx \, ds. \tag{1.7}$$

provided $0 \leqslant \phi \in C_0^\infty(\mathbb{R}^3 \times (0, t])$.

It should be noted that the following cancellation

$$\int_0^t \int_{\mathbb{R}^3} S_\alpha [\mathbf{d}_1, \mathbf{d}_2] : \nabla \mathbf{u} \phi \, dx \, ds = \int_0^t \int_{\mathbb{R}^3} T_\alpha [\nabla \mathbf{u}, \mathbf{d}_2] \cdot \mathbf{d}_1 \phi \, dx \, ds \tag{1.8}$$

plays a critical role in the later analysis.

Definition. A weak solution $(\mathbf{u}, \mathbf{d}, P) \in (L_t^\infty L_x^2 \cap L_t^2 H_x^1)(\mathbb{R}^3 \times (0, \infty), \mathbb{R}^3) \times (L_t^\infty H_x^1 \cap L_t^2 H_x^2)(\mathbb{R}^3 \times (0, \infty), \mathbb{R}^3) \times L_x^{\frac{5}{3}}(\mathbb{R}^3 \times (0, \infty))$ of (1.2) is a suitable weak solution of (1.2), if in addition, $(\mathbf{u}, \mathbf{d}, P)$ satisfies the local energy inequalities (1.7).

The main theorem of this paper concerns both the existence and partial regularity of suitable weak solutions to the simplified Ericksen–Leslie model.

Theorem 1.1. *For any* $\mathbf{u}_0 \in \mathbf{H}, \mathbf{d}_0 \in H^1(\mathbb{R}^3, \mathbb{R}^3)$, *there exists a global suitable weak solution* $(\mathbf{u}, \mathbf{d}, P) : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$ *of the simplified Ericksen–Leslie system* (1.2) *and* (1.3) *such that*

$$(\mathbf{u}, \mathbf{d}) \in C^{\infty}(\mathbb{R}^3 \times (0, \infty) \setminus \Sigma),$$

where $\Sigma \subset \mathbb{R}^3 \times \mathbb{R}_+$ *is a closed subset with* $\mathcal{P}^{\frac{15}{7}+\sigma}(\Sigma) = 0, \ \forall \ \sigma > 0.$

A couple of remarks on the presence and size of the singular set Σ are in orders.

Remark 1.2. Mathematically, it is a very challenging problem to ask if the set of singularity Σ is empty or not. Physically, the presence of potential singular set Σ for a solution $(\mathbf{u}, \mathbf{d})$ to the hydrodynamic system (1.2) may arise from the 3D turbulence phenomenons of the underlying fluids (e.g., vortex points, lines, or filaments) as well as the defects of the liquid crystal molecular alignment field $\mathbf{d}$ induced by the rotating and stretching effects of fluid velocity field $\mathbf{u}$, see for example Chorin [4]. While Mandelbrot conjectured in [22, 23] that the self-similar nature of turbulence of the fluid may result in concentration of possible singularities of $\mathbf{u}$ on a set of fractional Hausdorff dimension.

Remark 1.3. The best known result on the set of singularities for the Navier–Stokes equation was obtained by Caffarelli–Kohn–Nirenberg [3], which asserts that it has zero one-dimensional parabolic Hausdorff measure. For the co-rotational Beris–Edward Q-tensor system for liquid crystals, a result similar to [3] was also obtained by [6]. While our estimate on the dimension, $\frac{15}{7}$, of the singular set Σ in theorem 1.1 may not be optimal, it is a natural consequence resulting from the blowup analysis (see lemma 4.1) and the fractional Sobolev space regularity of the director field, i.e. $\mathbf{d} \in W_{\frac{20}{7}}^{1;\frac{1}{2}}(Q_T)$ (see the section 5 below).

This paper is organized as follows. In section 2, we will derive both the global and local energy inequality for smooth solutions of (1.2) and (1.3). In section 3, we will demonstrate the construction of suitable weak solution. In section 4, we will prove the ε_0-regularity criteria for the suitable weak solutions. In section 5, we will finish the proof of the theorem 1.1.

2. Global and local energy inequalities

In this section, we will derive both the global and local energy equalities for smooth solutions to (1.2).

Lemma 2.1. *Let* $(\mathbf{u}, \mathbf{d}) \in C^{\infty}(\mathbb{R}^3 \times [0, \infty), \mathbb{R}^3 \times \mathbb{R}^3)$ *be a solution to the simplified Ericksen–Leslie system* (1.2). *Then it holds that*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \frac{1}{2} \left(|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2 \right) + F(\mathbf{d})\mathrm{d}x + \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 + |\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})|^2 \mathrm{d}x = 0. \quad (2.1)$$

Proof. The proof is standard. See for instance [24, 25]. □

Lemma 2.2. *Let* $(\mathbf{u}, \mathbf{d}, P) \in C^{\infty}(\mathbb{R}^3 \times (0, \infty), \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R})$ *be a solution to* (1.2). *Then for all* $0 \leqslant \phi \in C_0^{\infty}(\mathbb{R}^3 \times (0, \infty))$, *it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \left[\frac{1}{2} \left(|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2 \right) + F(\mathbf{d}) \right] \phi \, \mathrm{d}x + \int_{\mathbb{R}^3} \left(|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d}|^2 + |\mathbf{f}(\mathbf{d})|^2 \right) \phi \mathrm{d}x$$

$$= \int_{\mathbb{R}^3} \left[\frac{1}{2}(|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(\partial_t \phi + \Delta \phi) + F(\mathbf{d})\partial_t \phi \right] \mathrm{d}x$$

$$+ \int_{\mathbb{R}^3} \left[\frac{1}{2} \left(|\mathbf{u}|^2 + 2P \right) \mathbf{u} \cdot \nabla \phi + \nabla \mathbf{d} \odot \nabla \mathbf{d} : \mathbf{u} \otimes \nabla \phi \right] \mathrm{d}x$$

$$+ \int_{\mathbb{R}^3} \left(\nabla \mathbf{d} \odot \nabla \mathbf{d} - |\nabla \mathbf{d}|^2 I_3 \right) : \nabla^2 \phi(x, s) \mathrm{d}x$$

$$+ \int_{\mathbb{R}^3} S_\alpha [\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}), \mathbf{d}] : (\mathbf{u} \otimes \nabla \phi)(x, s) \mathrm{d}x$$

$$+ \int_{\mathbb{R}^3} T_\alpha [\nabla \mathbf{u}, \mathbf{d}] \cdot (\nabla \phi \cdot \nabla \mathbf{d})(x, s) \mathrm{d}x$$

$$- \int_{\mathbb{R}^3} \mathbf{f}(\mathbf{d}) \cdot (\nabla \phi \cdot \nabla \mathbf{d}) \mathrm{d}x - 2 \int_{\mathbb{R}^3} \nabla \mathbf{f}(\mathbf{d}) : \nabla \mathbf{d} \phi(x, s) \mathrm{d}x. \tag{2.2}$$

Proof. Multiplying the **u** equation in (1.2) by $\mathbf{u}\phi$, integrating over $\mathbb{R}^3$, and by integration by parts we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \frac{1}{2} |\mathbf{u}|^2 \phi \, \mathrm{d}x + \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 \phi \, \mathrm{d}x$$

$$= \int_{\mathbb{R}^3} \left[\frac{1}{2} |\mathbf{u}|^2 (\partial_t \phi + \Delta \phi) + \frac{1}{2} (|\mathbf{u}|^2 + 2P) \mathbf{u} \cdot \nabla \phi \right] \mathrm{d}x$$

$$- \int_{\mathbb{R}^3} (\mathbf{u} \cdot \nabla \mathbf{d}) \cdot \Delta \mathbf{d} \phi \, \mathrm{d}x + \int_{\mathbb{R}^3} (\mathbf{u} \cdot \nabla \mathbf{d}) \cdot \mathbf{f}(\mathbf{d}) \phi \mathrm{d}x$$

$$+ \int_{\mathbb{R}^3} S_\alpha [\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}), \mathbf{d}] : (\mathbf{u} \otimes \nabla \phi) \, \mathrm{d}x + \int_{\mathbb{R}^3} S_\alpha [\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}), \mathbf{d}] : \nabla \mathbf{u} \phi \, \mathrm{d}x \tag{2.3}$$

By taking derivatives of **d** equation in (1.2), we have

$$\partial_t \nabla \mathbf{d} + \nabla (\mathbf{u} \cdot \nabla \mathbf{d}) = \nabla (\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}) + T_\alpha [\nabla \mathbf{u}, \mathbf{d}]).$$

Then multiplying this equation by $\nabla \mathbf{d} \phi$, integrating over $\mathbb{R}^3$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \frac{1}{2} |\nabla \mathbf{d}|^2 \phi \, \mathrm{d}x + \int_{\mathbb{R}^3} |\Delta \mathbf{d}|^2 \phi \, \mathrm{d}x$$

$$= \int_{\mathbb{R}^3} \frac{1}{2} |\nabla \mathbf{d}|^2 \partial_t \phi + \int_{\mathbb{R}^3} (\mathbf{u} \cdot \nabla \mathbf{d}) \cdot (\Delta \mathbf{d} \phi + \nabla \phi \cdot \nabla \mathbf{d}) \mathrm{d}x$$

$$- \int_{\mathbb{R}^3} \Delta \mathbf{d} \cdot (\nabla \phi \cdot \nabla \mathbf{d}) \mathrm{d}x - \int_{\mathbb{R}^3} \nabla (\mathbf{f}(\mathbf{d})) : \nabla \mathbf{d} \phi \, \mathrm{d}x$$

$$- \int_{\mathbb{R}^3} T_\alpha [\nabla \mathbf{u}, \mathbf{d}] \cdot (\nabla \phi \cdot \nabla \mathbf{d}) \mathrm{d}x - \int_{\mathbb{R}^3} T_\alpha [\nabla \mathbf{u}, \mathbf{d}] \cdot \Delta \mathbf{d} \phi \, \mathrm{d}x. \tag{2.4}$$

It follows from direct calculations that

$$- \int_{\mathbb{R}^3} \Delta \mathbf{d} \cdot (\nabla \phi \cdot \nabla \mathbf{d}) \mathrm{d}x = \int_{\mathbb{R}^3} \frac{1}{2} |\nabla \mathbf{d}|^2 \Delta \phi \, \mathrm{d}x + \int_{\mathbb{R}^3} (\nabla \mathbf{d} \odot \nabla \mathbf{d} - |\nabla \mathbf{d}|^2 I_3) : \nabla^2 \phi \, \mathrm{d}x.$$

$$\tag{2.5}$$

Moreover, multiplying the **d** equations by $\mathbf{f}(\mathbf{d})\phi$, integrating over $\mathbb{R}^3$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} F(\mathbf{d})\phi\,\mathrm{d}x + \int |\mathbf{f}(\mathbf{d})|^2\phi\,\mathrm{d}x = \int_{\mathbb{R}^3} (F(\mathbf{d})\partial_t\phi - (\mathbf{u}\cdot\nabla\mathbf{d})\cdot\mathbf{f}(\mathbf{d})\phi)\mathrm{d}x$$

$$+ \int_{\mathbb{R}^3} T_\alpha[\nabla\mathbf{u},\mathbf{d}]\cdot\mathbf{f}(\mathbf{d})\phi\,\mathrm{d}x - \int_{\mathbb{R}^3}(\nabla\mathbf{f}(\mathbf{d}):\nabla\mathbf{d}\phi + (\nabla\phi\cdot\nabla\mathbf{d})\cdot\mathbf{f}(\mathbf{d}))\mathrm{d}x. \qquad (2.6)$$

Hence, by adding (2.3), (2.4) and (2.5) together, and applying (1.8), we get (1.7). □

3. Existence of suitable weak solutions

In this section, we will follow the same scheme in [3, 6] to construct a suitable weak solution to (1.2).

We introduce the so-called retarded mollifier Ψ_θ for $f : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}$, with $0 < \theta < 1$,

$$\Psi_\theta[f](x,t) = \frac{1}{\theta^4}\int_{\mathbb{R}^4}\eta\left(\frac{y}{\theta},\frac{\tau}{\theta}\right)\tilde{f}(x-y,t-\tau)\mathrm{d}y\,\mathrm{d}\tau,$$

where

$$\tilde{f}(x,t) = \begin{cases} f(x,t) & t \geqslant 0, \\ 0 & t < 0, \end{cases}$$

and the mollifying function $\eta \in C_0^\infty(\mathbb{R}^4)$ satisfies

$$\begin{cases} \eta \geqslant 0 \quad \text{and} \quad \int_{\mathbb{R}^4}\eta\,\mathrm{d}x\,\mathrm{d}t = 1, \\ \operatorname{spt}\eta \subset \left\{(x,t) : |x|^2 < t, 1 < t < 2\right\}. \end{cases}$$

It is easy to verify that for $\theta \in (0,1]$ and $0 < T \leqslant \infty$ that

$$\operatorname{div}\Psi_\theta[\mathbf{u}] = 0 \quad \text{if } \operatorname{div}\mathbf{u} = 0,$$

$$\sup_{0\leqslant t\leqslant T}\int_{\mathbb{R}^3}|\Psi_\theta[\mathbf{w}]|^2(x,t)\mathrm{d}x \leqslant C\sup_{0\leqslant t\leqslant T}\int_{\mathbb{R}^3}|\mathbf{w}|^2(x,t)\mathrm{d}x,$$

$$\int_{\mathbb{R}^3\times[0,T]}|\nabla\Psi_\theta[\mathbf{w}]|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \leqslant C\int_{\mathbb{R}^3\times[0,T]}|\nabla\mathbf{w}|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Now with the mollifier $\Psi_\theta[\mathbf{w}] \in C^\infty(\mathbb{R}^4)$, we introduce the approximate system of (1.2):

$$\begin{cases} \partial_t\mathbf{u}^\theta + \Psi_\theta[\mathbf{u}^\theta]\cdot\nabla\mathbf{u}^\theta + \nabla P^\theta = \Delta\mathbf{u}^\theta - \nabla\Psi_\theta[\mathbf{d}^\theta]\cdot(\Delta\mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta)) \\ \qquad -\nabla\cdot S_\alpha[\Delta\mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta), \Psi_\theta[\mathbf{d}^\theta]], \\ \nabla\cdot\mathbf{u}^\theta = 0, & \text{in } \mathbb{R}^3\times(0,T) \\ \partial_t\mathbf{d}^\theta + \mathbf{u}^\theta\cdot\nabla\Psi_\theta[\mathbf{d}^\theta] - T_\alpha[\nabla\mathbf{u}^\theta, \Psi_\theta[\mathbf{d}^\theta]] = \Delta\mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta). \end{cases}$$

$$(3.1)$$

subject to the initial and boundary condition (1.3).

For a fixed large integer $N \geqslant 1$, set $\theta = \frac{T}{N} \in (0, 1]$, we want to find $(\mathbf{u}^\theta, \mathbf{d}^\theta, P^\theta)$ solving (3.1). This amounts to solving a coupling system of a Stokes-like system for $\mathbf{u}$ and a semi-linear parabolic-like equation for $\mathbf{d}$ with smooth coefficients. For $t \in [0, \theta]$, we have $\Psi_\theta[\mathbf{u}^\theta] = \Psi_\theta[\mathbf{d}^\theta] = 0$, and the system (3.1) reduces to a decoupled system

$$\begin{cases} \partial_t \mathbf{u}^\theta + \nabla P^\theta = \Delta \mathbf{u}^\theta, \\[6pt] \nabla \cdot \mathbf{u}^\theta = 0, \\[6pt] \partial_t \mathbf{d}^\theta = \Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta), \\[6pt] (\mathbf{u}^\theta, \mathbf{d}^\theta)\big|_{t=0} = (\mathbf{u}_0, \mathbf{d}_0) \end{cases} \quad \text{in } \mathbb{R}^3 \times [0, \theta]. \tag{3.2}$$

which can be solved easily by the standard theory. Suppose now that the (3.1) has been solved in $t \in [0, k\theta]$ for some $0 \leqslant k < N - 1$. We are going to solve (3.1) in the time interval $[k\theta, (k+1)\theta]$ with an initial data

$$(\mathbf{u}, \mathbf{d})\big|_{t=k\theta} = \lim_{t \uparrow k\theta} (\mathbf{u}^\theta, \mathbf{d}^\theta)(\cdot, t) \quad \text{in } \mathbb{R}^3. \tag{3.3}$$

Then one can solve the coupling system (3.1) using the Faedo–Galerkin method. In fact, for a pair of smooth test functions $(\phi, \psi) \in \mathbf{V} \times H^2(\mathbb{R}^3, \mathbb{R}^3)$, the weak formulation for (3.1) reads

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \mathbf{u}^\theta \cdot \phi \, \mathrm{d}x + \int_{\mathbb{R}^3} (\Psi_\theta[\mathbf{u}^\theta] \cdot \nabla \mathbf{u}^\theta) \cdot \phi \, \mathrm{d}x + \int_{\mathbb{R}^3} \nabla \mathbf{u}^\theta : \nabla \phi \, \mathrm{d}x$$

$$= -\int_{\mathbb{R}^3} (\phi \cdot \nabla \Psi_\theta[\mathbf{d}^\theta]) \cdot (\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta)) \mathrm{d}x + \int_{\mathbb{R}^3} S_\alpha[\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta), \Psi_\theta[\mathbf{d}^\theta]] : \nabla \phi \, \mathrm{d}x, \tag{3.4}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \nabla \mathbf{d}^\theta : \nabla \psi \, \mathrm{d}x - \int_{\mathbb{R}^3} (\mathbf{u}^\theta \cdot \nabla \Psi_\theta[\mathbf{d}^\theta]) \cdot \Delta \psi \, \mathrm{d}x$$

$$= -\int_{\mathbb{R}^3} (\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta)) \cdot \Delta \psi \, \mathrm{d}x - \int_{\mathbb{R}^3} T_\alpha[\nabla \mathbf{u}^\theta, \Psi_\theta[\mathbf{d}^\theta]] \cdot \Delta \psi \, \mathrm{d}x. \tag{3.5}$$

We can solve the ODE system (3.4)–(3.5) with test function (ψ, ϕ) chosen to be the basis of $\mathbf{V} \times H^2(\mathbb{R}^3, \mathbb{R}^3)$ up to a short time interval $[k\theta, k\theta + T_0]$. Multiplying the $\mathbf{u}^\theta$ equation in (3.1) by $\mathbf{u}^\theta$, and the $\mathbf{d}^\theta$ equation by $-\Delta \mathbf{d}^\theta + \mathbf{f}(\mathbf{d}^\theta)$, integrating over $\mathbb{R}^3$ and adding two equations together we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \frac{1}{2} \left(|\mathbf{u}^\theta|^2 + |\nabla \mathbf{d}^\theta|^2 \right) + F(\mathbf{d}^\theta) \mathrm{d}x + \int_{\mathbb{R}^3} \left(|\nabla \mathbf{u}^\theta|^2 + |\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta)|^2 \right) \mathrm{d}x = 0. \tag{3.6}$$

Next we need a uniform bound on $(\mathbf{u}^\theta, \mathbf{d}^\theta, P^\theta)$ to pass the limit $\theta \to 0$ to get a suitable weak solution. First by direct calculations we can show that

$$\int_{\mathbb{R}^3} |\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta)|^2 \mathrm{d}x = \int_{\mathbb{R}^3} [|\Delta \mathbf{d}^\theta|^2 + |\mathbf{f}(\mathbf{d}^\theta)|^2 - 2\Delta \mathbf{d}^\theta \cdot \mathbf{f}(\mathbf{d}^\theta)] \mathrm{d}x$$

$$= \int_{\mathbb{R}^3} (|\Delta \mathbf{d}^\theta|^2 + |\mathbf{f}(\mathbf{d}^\theta)|^2 + 2\nabla \mathbf{d}^\theta : \nabla \mathbf{f}(\mathbf{d}^\theta)) \mathrm{d}x$$

$$= \int_{\mathbb{R}^3} (|\Delta \mathbf{d}^\theta|^2 + |\mathbf{f}(\mathbf{d}^\theta)|^2 - 2|\nabla \mathbf{d}^\theta|^2 + 2|\nabla \mathbf{d}^\theta|^2 |\mathbf{d}^\theta|^2$$
$$+ 4|(\nabla \mathbf{d}^\theta)^T \mathbf{d}^\theta|^2) \mathrm{d}x. \tag{3.7}$$

From (3.6), we can obtain that

$$\sup_{0 < \theta < 10} \sup_{0 < t < T} \int_{\mathbb{R}^3} \left(|\mathbf{u}^\theta|^2 + |\nabla \mathbf{d}^\theta|^2 \right) \mathrm{d}x + \int_0^T \int_{\mathbb{R}^3} |\nabla \mathbf{u}^\theta|^2$$
$$+ |\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta)|^2 \mathrm{d}x \, \mathrm{d}t \leqslant C(\mathbf{u}_0, \mathbf{d}_0). \tag{3.8}$$

Combining (3.7) and (3.8), we get

$$\int_0^T \int_{\mathbb{R}^3} |\Delta \mathbf{d}^\theta|^2 + |\mathbf{f}(\mathbf{d}^\theta)|^2 \mathrm{d}x \, \mathrm{d}t \leqslant \int_0^T \int_{\mathbb{R}^3} |\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta)|^2 \mathrm{d}x \, \mathrm{d}t + 2 \int_0^T \int_{\mathbb{R}^3} |\nabla \mathbf{d}^\theta|^2 \mathrm{d}x \, \mathrm{d}t$$
$$\leqslant \int_0^T \int_{\mathbb{R}^3} |\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta)|^2 \mathrm{d}x \, \mathrm{d}t + 2T \sup_{0 < t < T} \int_{\mathbb{R}^3} |\nabla \mathbf{d}^\theta|^2 \mathrm{d}x$$
$$\leqslant C(\mathbf{u}_0, \mathbf{d}_0, T), \tag{3.9}$$

From (3.8) and (3.9), we have that $\mathbf{u}^\theta$ is uniformly bounded in $L_t^2 H_x^1(\mathbb{R}^3 \times [0, T])$, $\mathbf{d}^\theta$ is uniformly bounded in $L_t^2 H_x^2(K \times [0, T])$ for any compact set $K \subset \mathbb{R}^3$, and $\nabla \mathbf{d}^\theta$ is uniformly bounded in $L_t^2 H_x^1(\mathbb{R}^3 \times [0, T])$. Therefore, after passing to a subsequence, there exist $\mathbf{u} \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(\mathbb{R}^3 \times [0, T])$, $\mathbf{d} \in \cap_{R>0} L_t^\infty L_x^4(B_R \times [0, T])$, $\nabla \mathbf{d} \in L_t^\infty L_x^2 \cap L_t^2 H_x^1$ $(\mathbb{R}^3 \times [0, T])$ such that

$$\begin{cases} \mathbf{u}^\theta \rightharpoonup \mathbf{u} & \text{in } L_t^\infty L_x^2 \cap L_t^2 H_x^1(\mathbb{R}^3 \times [0, T]), \\ \nabla \mathbf{d}^\theta \rightharpoonup \nabla \mathbf{d} & \text{in } L_t^\infty L_x^2 \cap L_t^2 H_x^1(\mathbb{R}^3 \times [0, T]), \\ \mathbf{f}(\mathbf{d}^\theta) \rightharpoonup \mathbf{f}(\mathbf{d}) & \text{in } L_t^2 L_x^2(\mathbb{R}^3 \times [0, T]). \end{cases} \tag{3.10}$$

By the Sobolev-interpolation inequality, we have that $\nabla \mathbf{d}^\theta \in L_t^{10} L_x^{\frac{30}{13}}$, $\mathbf{d}^\theta \in L_t^{10} L_x^{10}$, and

$$\int_0^T \left\| \nabla \mathbf{d}^\theta \right\|_{L_x^{\frac{30}{13}}}^{10} \mathrm{d}t \leqslant \int_0^T \left\| \nabla \mathbf{d}^\theta \right\|_{L_x^2}^8 \left\| \nabla \mathbf{d}^\theta \right\|_{L_x^6}^2 \mathrm{d}t \leqslant \left\| \mathbf{d}^\theta \right\|_{L_t^\infty H_x^1}^8 \left\| \mathbf{d}^\theta \right\|_{L_t^2 H_x^2}^2 < \infty,$$
$$\int_0^T \left\| \mathbf{d}^\theta \right\|_{L_x^{10}}^{10} \mathrm{d}t \leqslant C \int_0^T \left\| \mathbf{d}^\theta \right\|_{W_x^{1, \frac{30}{13}}}^{10} \mathrm{d}t < \infty. \tag{3.11}$$

By the lower semicontinuity and (3.6), we have, for $E(\mathbf{u}, \mathbf{d}) = \int_{\mathbb{R}^3} \frac{1}{2}(|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2 + F(\mathbf{d})) \mathrm{d}x$, that

$$E(\mathbf{u}, \mathbf{d})(t) + \int_0^t \int_{\mathbb{R}^3} (|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})|^2) \, \mathrm{d}x \, \mathrm{d}t \leqslant E(\mathbf{u}_0, \mathbf{d}_0) \tag{3.12}$$

holds for a.e. $0 \leqslant t \leqslant T$.

Now we want to estimate the pressure function P^θ. Taking the divergence of $\mathbf{u}^\theta$ equation in (3.1) gives

$$-\Delta P^\theta = \mathrm{div}^2(\Psi_\theta[\mathbf{u}^\theta] \otimes \mathbf{u}^\theta) + \mathrm{div}\left(\nabla(\Psi_\theta[\mathbf{d}^\theta]) \cdot (\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta))\right)$$
$$+ \mathrm{div}^2\left[S_\alpha[\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta), \Psi_\theta[\mathbf{d}^\theta]]\right], \quad \text{in } \mathbb{R}^3. \tag{3.13}$$

For P^θ, we claim that P^θ in $L^{\frac{5}{3}}(\mathbb{R}^3 \times [0, T])$ and

$$\left\| P^\theta \right\|_{L^{\frac{5}{3}}(\mathbb{R}^3 \times [0,T])} \leqslant C(\|u_0\|_{L^2(\mathbb{R}^3)}, \|\mathbf{d}_0\|_{H^1(\mathbb{R}^3)}, T), \ \forall \ \theta \in (0, 1].$$

In fact, by Calderon–Zgymund's L^p-theory, we have

$$\left\| P^\theta \right\|_{L^{\frac{5}{3}}(\mathbb{R}^3 \times [0,T])}$$

$$\leqslant C \left[\left\| \Psi_\theta[\mathbf{u}^\theta] \otimes \mathbf{u}^\theta \right\|_{L_t^{\frac{5}{3}} L_x^{\frac{5}{3}}} + \left\| \nabla(\Psi_\theta[\mathbf{d}^\theta]) \cdot (\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta)) \right\|_{L_t^{\frac{5}{3}} L_x^{\frac{15}{14}}} \right.$$

$$\left. + \left\| |\Psi_\theta[\mathbf{d}^\theta]| |\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta)| \right\|_{L_t^{\frac{5}{3}} L_x^{\frac{5}{3}}} \right]$$

$$\leqslant C \left[\left\| \mathbf{u}^\theta \right\|^2_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}} + \left\| \nabla \mathbf{d}^\theta \right\|_{L_t^{10} L_x^{\frac{30}{13}}} \left\| \Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta) \right\|_{L_t^2 L_x^2} \right.$$

$$\left. + \left\| \mathbf{d}^\theta \right\|_{L_t^{10} L_x^{10}} \left\| \Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta) \right\|_{L_t^2 L_x^2} \right]$$

$$\leqslant C(\|\mathbf{u}\|_{L_t^\infty L_x^2 \cap L_t^2 H_x^1(\mathbb{R}^3 \times [0,T])}, \|\mathbf{d}\|_{L_t^\infty H_x^1 \cap L_t^2 H_x^2(\mathbb{R}^3 \times [0,T])})$$

$$\leqslant C(\|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)}, \|\mathbf{d}_0\|_{H^1(\mathbb{R}^3)}, T).$$

This uniform estimate implies that there exists $P \in L^{\frac{5}{3}}(\mathbb{R}^3 \times [0, T])$ such that as $\theta \to 0$,

$$P^\theta \rightharpoonup P \quad \text{in } L^{\frac{5}{3}}(\mathbb{R}^3 \times [0, T]). \tag{3.14}$$

Recalling the $\mathbf{u}^\theta$ equation, we get

$$\partial_t \mathbf{u}^\theta = -\Psi_\theta[\mathbf{u}^\theta] \cdot \nabla \mathbf{u}^\theta - \nabla P^\theta + \Delta \mathbf{u}^\theta - \nabla(\Psi_\theta[\mathbf{d}^\theta]) \cdot (\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta))$$

$$- \nabla \cdot S_\alpha[\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}^\theta), \Psi_\theta[\mathbf{d}^\theta]]$$

$$\in L^{\frac{5}{4}}(\mathbb{R}^3 \times [0, T]) + L^{\frac{5}{3}}([0, T], W^{-1, \frac{5}{3}}(\mathbb{R}^3)) + \bigcap_{R > 0} L^2([0, T], W^{-1, \frac{3}{2}}(B_R)),$$

and

$$\sup_{0 < \theta < 1} \left\| \partial_t \mathbf{u}^\theta \right\|_{L^{\frac{5}{4}}(\mathbb{R}^3 \times [0,T]) + L^{\frac{5}{3}}([0,T], W^{-1,\frac{5}{3}}(\mathbb{R}^3)) + L^2([0,T], W^{-1,\frac{3}{2}}(B_R))}$$

$$\leqslant C(R, T, \|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)}, \|\mathbf{d}_0\|_{H^1(\mathbb{R}^3)}).$$

Similarly, we can show

$$\partial_t \mathbf{d}^\theta \in L^{\frac{5}{3}}(\mathbb{R}^3 \times [0, T]) + \bigcap_{R > 0} L^2([0, T], L^{\frac{4}{3}}(B_R)),$$

and

$$\left\| \partial_t \mathbf{d}^\theta \right\|_{L^{\frac{5}{3}}(\mathbb{R}^3 \times [0,T]) + \bigcap_{R>0} L^2([0,T], L^{\frac{3}{2}}(B_R))(\mathbb{R}^3 \times [0,T])}$$

$$\leqslant C(R, T, \|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)}, \|\mathbf{d}_0\|_{H^1(\mathbb{R}^3)}).$$

Hence by the Sobolev embedding and Aubin–Lions' compactness lemma, we can conclude that as $\theta \to 0$,

$$
\begin{cases}
\mathbf{u}^\theta \to \mathbf{u} & \text{in } L^{p_1}(\mathbb{R}^3 \times [0,T]), 1 < p_1 < \dfrac{10}{3}, \\[2mm]
\nabla \mathbf{u}^\theta \rightharpoonup \nabla \mathbf{u} & \text{in } L^2(\mathbb{R}^3 \times [0,T]), \\[2mm]
\mathbf{d}^\theta \to \mathbf{d} & \text{in } L^{p_2}(\mathbb{R}^3 \times [0,T]), 1 < p_2 < 10, \\[2mm]
\nabla \mathbf{d}^\theta \to \nabla \mathbf{d} & \text{in } L^{p_1}(\mathbb{R}^3 \times [0,T]), 1 < p_1 < \dfrac{10}{3}, \\[2mm]
\nabla^2 \mathbf{d}^\theta \rightharpoonup \nabla^2 \mathbf{d} & \text{in } L^2(\mathbb{R}^3 \times [0,T]).
\end{cases}
\tag{3.15}
$$

Furthermore, $(\mathbf{u}^\theta, \mathbf{d}^\theta, P^\theta)$ satisfies the local energy inequality. In fact, if we multiply the $\mathbf{u}^\theta$ equation in (3.1) by $\mathbf{u}^\theta \phi$, take derivative of the $\mathbf{d}^\theta$ equation in (3.1) and multiply by $\nabla \mathbf{d}^\theta \phi$, multiply the $\mathbf{d}^\theta$ equation in (3.1) by $\mathbf{f}(\mathbf{d}^\theta)$, and perform calculations similar to the previous section, we can get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \left[\frac{1}{2}\left(|\mathbf{u}^\theta|^2 + |\nabla \mathbf{d}^\theta|^2\right) + F(\mathbf{d}^\theta) \right] \phi \, \mathrm{d}x + \int_{\mathbb{R}^3} \left(|\nabla \mathbf{u}^\theta|^2 + |\Delta \mathbf{d}^\theta|^2 + |\mathbf{f}(\mathbf{d}^\theta)|^2 \right) \phi \, \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \left[\frac{1}{2}(|\mathbf{u}^\theta|^2 + |\nabla \mathbf{d}^\theta|^2)(\partial_t \phi + \Delta \phi) + F(\mathbf{d}^\theta)\partial_t \phi \right] \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3} \left[\frac{1}{2}|\mathbf{u}^\theta|^2 \Psi_\theta[\mathbf{u}^\theta] \cdot \nabla \phi + P^\theta \mathbf{u}^\theta \cdot \nabla \phi + \nabla \Psi_\theta[\mathbf{d}^\theta] \odot \nabla \mathbf{d}^\theta : \mathbf{u}^\theta \otimes \nabla \phi \right] \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3} \left(\nabla \mathbf{d}^\theta \odot \nabla \mathbf{d}^\theta - |\nabla \mathbf{d}^\theta|^2 I_3 \right) : \nabla^2 \phi \, \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3} S_\alpha[\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta), \Psi_\theta[\mathbf{d}^\theta]] : (\mathbf{u}^\theta \otimes \nabla \phi)\mathrm{d}x + \int_{\mathbb{R}^3} T_\alpha[\nabla \mathbf{u}^\theta, \Psi_\theta[\mathbf{d}^\theta]] \cdot (\nabla \phi \cdot \nabla \mathbf{d}^\theta)\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3} \mathbf{f}(\mathbf{d}^\theta) \cdot (\nabla \phi \cdot \nabla \mathbf{d}^\theta)\mathrm{d}x - 2\int_{\mathbb{R}^3} \nabla \mathbf{f}(\mathbf{d}^\theta) : \nabla \mathbf{d}^\theta \phi \, \mathrm{d}x.
\end{aligned}
\tag{3.16}
$$

With the convergence (3.14), (3.15), it is easy to check that the limit $(\mathbf{u}, \mathbf{d})$ is a weak solution to (1.2) and (1.3). Taking the limit in (3.16) as $\theta \to 0$, by the lower semicontinuity we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \left[\frac{1}{2}(|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2) + F(\mathbf{d}) \right] \phi(x,t)\mathrm{d}x + \int_0^t \int_{\mathbb{R}^3} (|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d}|^2 + |\mathbf{f}(\mathbf{d})|^2)\phi \, \mathrm{d}x \, \mathrm{d}s \\
&\leqslant \liminf_{\theta \to 0} \int_{\mathbb{R}^3} \left[\frac{1}{2}(|\mathbf{u}^\theta|^2 + |\nabla \mathbf{d}^\theta|^2) + F(\mathbf{d}^\theta) \right] \phi(x,t)\mathrm{d}x \\
&\quad + \int_0^t \int_{\mathbb{R}^3} (|\nabla \mathbf{u}^\theta|^2 + |\Delta \mathbf{d}^\theta|^2 + |\mathbf{f}(\mathbf{d}^\theta)|^2)\phi \mathrm{d}x \, \mathrm{d}s \Bigg].
\end{aligned}
\tag{3.17}
$$

While

$$\lim_{\theta \to 0} \text{R.H.S. of (3.16)}$$

$$= \int_{\mathbb{R}^3} \frac{1}{2}(|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(\partial_t \phi + \Delta \phi) + F(\mathbf{d})\partial_t \phi \, \mathrm{d}x$$

$$+ \int_{\mathbb{R}^3} \left[\frac{1}{2}\left(|\mathbf{u}|^2 + 2P\right) \mathbf{u} \cdot \nabla \phi + \nabla \mathbf{d} \odot \nabla \mathbf{d} : \mathbf{u} \otimes \nabla \phi \right] \mathrm{d}x$$

$$+ \int_{\mathbb{R}^3} \left(\nabla \mathbf{d} \odot \nabla \mathbf{d} - |\nabla \mathbf{d}|^2 I_3 \right) : \nabla^2 \phi \, \mathrm{d}x \qquad (3.18)$$

$$+ \int_{\mathbb{R}^3} S_\alpha[\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}), \mathbf{d}] : (\mathbf{u} \otimes \nabla \phi)\mathrm{d}x$$

$$+ \int_{\mathbb{R}^3} T_\alpha[\nabla \mathbf{u}, \mathbf{d}] \cdot (\nabla \phi \cdot \nabla \mathbf{d})\mathrm{d}x$$

$$- \int_{\mathbb{R}^3} \mathbf{f}(\mathbf{d}) \cdot (\nabla \phi \cdot \nabla \mathbf{d})\mathrm{d}x - 2 \int_{\mathbb{R}^3} \nabla \mathbf{f}(\mathbf{d}) : \nabla \mathbf{d} \phi \, \mathrm{d}x.$$

Putting all those together we show that the local energy inequality (1.7) holds. Therefore $(\mathbf{u}, \mathbf{d}, P)$ is a suitable weak solution to (1.2) and (1.3).

4. ε_0-Regularity criteria

In this section we will establish the partial regularity for suitable weak solutions $(\mathbf{u}, \mathbf{d}, P)$ of (1.2) in $\mathbb{R}^3 \times (0, \infty)$. The argument is based on a blowing up argument, motivated by that of Lin [18] on the Navier–Stokes equation. Recently, this type of argument has been employed by Du–Hu–Wang [6] for the partial regularity in the co-rotational Beris–Edwards system in dimension three. However, the kinematic transport effects in (1.2) destroy the maximum principle for $\mathbf{d}$, which is necessary to apply the argument by [6, 18]. To overcome this new difficulty, we adapt some ideas from Giaquinta–Giusti [10] to control the mean oscillation of $\mathbf{d}$ in L^6. More precisely, we have

Lemma 4.1. *For any $M > 0$, there exist $\varepsilon_0 = \varepsilon_0(M) > 0$, $0 < \tau_0(M) < \frac{1}{2}$, and $C_0 = C_0$ $(M) > 0$, such that if $(\mathbf{u}, \mathbf{d}, P)$ is a suitable weak solution of (1.2) in $\mathbb{R}^3 \times (0, \infty)$, which satisfies, for $z_0 = (x_0, t_0) \in \mathbb{R}^3 \times (r^2, \infty)$ and $r > 0$,*

$$|\mathbf{d}_{z_0,r}| := \left| \fint_{\mathbb{P}_r(z_0)} \mathbf{d} \, \mathrm{d}x \, \mathrm{d}t \right| \leqslant M, \qquad (4.1)$$

and

$$\Phi(z_0, r) := r^{-2} \int_{\mathbb{P}_r(z_0)} \left(|\mathbf{u}|^3 + |\nabla \mathbf{d}|^3 \right) \mathrm{d}x \, \mathrm{d}t + \left(r^{-3} \int_{\mathbb{P}_r(z_0)} |P|^{\frac{3}{2}} \mathrm{d}x \, \mathrm{d}t \right)^2$$

$$+ \left(\fint_{\mathbb{P}_r(z_0)} |\mathbf{d} - \mathbf{d}_{z_0,r}|^6 \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{2}} \leqslant \varepsilon_0^3, \qquad (4.2)$$

then

$$\Phi(z_0, \tau_0 r) \leqslant \frac{1}{2} \max \left\{ \Phi(z_0, r), C_0 r^3 \right\}. \qquad (4.3)$$

Remark 4.2.   In the absence of maximum principle for the director field **d**, the L^6-norm of the mean oscillation of **d** plays the role in obtaining the (local) boundedness of $(\mathbf{u}, \nabla \mathbf{d}) \in L_t^\infty L_x^2 \cap L_t^2 H_x^1$ in (4.15). By closely examining the proof of lemma 4.1, the L^6-norm can be relaxed to the L^p-norm of the mean oscillation of **d** as long as $p > 5$. However, this does not seem to improve the estimate of the dimension of the singular set Σ of $(\mathbf{u}, \nabla \mathbf{d})$, since we can only obtain $\mathbf{d} \in W_{\frac{20}{7}}^{1,\frac{1}{2}}$, which can yield the boundedness of $L^{\frac{20}{3}}$-norm of the mean oscillation of **d** (see (5.4) below).

Proof.   We prove it by contradiction. Suppose that the conclusion were false. Then there exists $M_0 > 0$ such that for any $\tau \in (0, \frac{1}{2})$, there exist $\varepsilon_i \to 0, C_i \to \infty$, and $r_i > 0$, and $z_i = (x_i, t_i) \in \mathbb{R}^3 \times (r_i^2, \infty)$ such that

$$|\mathbf{d}_{z_i, r_i}| \leqslant M_0, \tag{4.4}$$

and

$$\Phi(z_i, r_i) = \varepsilon_i^3, \tag{4.5}$$

but

$$\Phi(z_i, \tau r_i) \geqslant \frac{1}{2} \max \left\{ \varepsilon_i^3, C_i r_i^3 \right\}. \tag{4.6}$$

Notice that

$$(\tau r_i)^{-2} \int_{\mathbb{P}_{\tau r_i}(z_i)} \left(|\mathbf{u}|^3 + |\nabla \mathbf{d}|^3 \right) \mathrm{d}x \, \mathrm{d}t + \left((\tau r_i)^{-2} \int_{\mathbb{P}_{\tau r_i}(z_i)} |P|^{\frac{3}{2}} \mathrm{d}x \, \mathrm{d}t \right)^2$$

$$\leqslant \tau^{-4} \left(r_i^{-2} \int_{\mathbb{P}_{r_i}(z_i)} \left(|\mathbf{u}|^3 + |\nabla \mathbf{d}|^3 \right) \mathrm{d}x \, \mathrm{d}t + \left(r_i^{-2} \int_{\mathbb{P}_{r_i}(z_i)} |P|^{\frac{3}{2}} \mathrm{d}x \, \mathrm{d}t \right)^2 \right),$$

$$\left(\fint_{\mathbb{P}_{\tau r_i}(z_i)} |\mathbf{d} - \mathbf{d}_{z_i, \tau r_i}|^6 \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{2}} \leqslant \left(2^5 |\mathbf{d}_{z_i, \tau r_i} - \mathbf{d}_{z_i, r_i}|^6 + 2^5 \fint_{\tau r_i(z_i)} |\mathbf{d} - \mathbf{d}_{z_i, r_i}|^6 \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{2}}$$

$$= \left(2^5 |\fint_{\mathbb{P}_{\tau r_i}(z_i)} (\mathbf{d} - \mathbf{d}_{z_i, r_i}) \mathrm{d}x \, \mathrm{d}t|^6 + 2^5 \fint_{\tau r_i(z_i)} |\mathbf{d} - \mathbf{d}_{z_i, r_i}|^6 \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{2}}$$

$$\leqslant \left(2^6 \fint_{\mathbb{P}_{\tau r_i}(z_i)} |\mathbf{d} - \mathbf{d}_{z_i, r_i}|^6 \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{2}} \leqslant 2^3 \tau^{-\frac{5}{2}} \left(\fint_{\mathbb{P}_{r_i}(z_i)} |\mathbf{d} - \mathbf{d}_{z_i, r_i}|^6 \right)^{\frac{1}{2}}.$$

From (4.6), we see that

$$C_i r_i^3 \leqslant 2\Phi(z_i, \tau r_i) \leqslant 2 \max \left\{ \tau^{-4}, 2^3 \tau^{-\frac{5}{2}} \right\} \Phi(z_i, r_i)$$

$$= 2 \max \left\{ \tau^{-4}, 2^3 \tau^{-\frac{5}{2}} \right\} \varepsilon_i^3$$

so that

$$r_i \leqslant \left(\frac{\varepsilon_i^3}{2C_i \max \left\{ \tau^{-4}, 2^3 \tau^{-\frac{5}{2}} \right\}} \right)^{\frac{1}{3}} \to 0. \tag{4.7}$$

Define the blowing-up sequence

$$(\mathbf{u}_i, \mathbf{d}_i, P_i) := \left(r_i \mathbf{u}, \mathbf{d}, r_i^2 P \right) (x_i + r_i x, t_i + r_i^2 t), \ \forall \ x \in \mathbb{R}^3, t > -1,$$

and

$$(\widehat{\mathbf{u}}_i, \widehat{\mathbf{d}}_i, \widehat{P}_i)(z) := \left(\frac{\mathbf{u}_i}{\varepsilon_i}, \frac{\mathbf{d}_i - \overline{\mathbf{d}}_i}{\varepsilon_i}, \frac{P_i}{\varepsilon_i} \right) (z), \ \forall \ z(x, t) \in \mathbb{P}_1(0),$$

where

$$\overline{\mathbf{d}}_i = \fint_{\mathbb{P}_1(0)} \mathbf{d}_i \, dx \, dt.$$

Then $(\widehat{\mathbf{u}}_i, \widehat{\mathbf{d}}_i, \widehat{P}_i)$ satisfies

$$\begin{cases} \fint_{\mathbb{P}_1(0)} \widehat{\mathbf{d}}_i \, dx \, dt = 0, \quad |\overline{\mathbf{d}}_i| = |\mathbf{d}_{z_i, r_i}| \leqslant M_0, \\[2mm] \int_{\mathbb{P}_1(0)} \left(|\widehat{\mathbf{u}}_i|^3 + |\nabla \widehat{\mathbf{d}}_i|^3 \right) dx \, dt + \left(\int_{\mathbb{P}_1(0)} |\widehat{P}_i|^{\frac{3}{2}} dx \, dt \right)^2 \\[2mm] + \left(\fint_{\mathbb{P}_1(0)} |\widehat{\mathbf{d}}_i|^6 dx \, dt \right)^{\frac{1}{2}} = 1, \\[2mm] \tau^{-2} \int_{\mathbb{P}_\tau(0)} \left(|\widehat{\mathbf{u}}_i|^3 + |\nabla \widehat{\mathbf{d}}_i|^3 \right) dx \, dt + \left(\tau^{-2} \int_{\mathbb{P}_\tau(0)} |\widehat{P}_i|^{\frac{3}{2}} dx \, dt \right)^2 \\[2mm] + \left(\fint_{\mathbb{P}_\tau(0)} |\widehat{\mathbf{d}}_i - (\widehat{\mathbf{d}}_i)_{0,\tau}|^6 dx \, dt \right)^{\frac{1}{2}} \geqslant \frac{1}{2} \max \left\{ 1, C_i \left(\frac{r_i}{\varepsilon_i} \right)^3 \right\}. \end{cases} \tag{4.8}$$

It follows from (4.4), (4.5) that

$$\begin{cases} \fint_{\mathbb{P}_1(0)} |\mathbf{d}_i|^6 dx \, dt \leqslant C \left(\fint_{\mathbb{P}_1(0)} |\mathbf{d}_i - \overline{\mathbf{d}}_i|^6 dx \, dt + |\overline{\mathbf{d}}_i|^6 \right) \leqslant C \left(\varepsilon_i^6 + M_0^6 \right), \\[2mm] \fint_{\mathbb{P}_1(0)} F(\mathbf{d}_i)^{\frac{3}{2}} dx \, dt \leqslant C \fint_{\mathbb{P}_1(0)} \big| |\mathbf{d}_i|^2 - 1 \big|^3 dx \, dt \leqslant C \left(\varepsilon_i^6 + M_0^6 + 1 \right), \\[2mm] \fint_{\mathbb{P}_1(0)} |\mathbf{f}(\mathbf{d}_i)|^2 dx \, dt \leqslant C \left(\fint_{\mathbb{P}_1(0)} |\mathbf{d}_i|^6 dx \, dt + 1 \right) \leqslant C(\varepsilon_i^6 + M_0^6 + 1), \\[2mm] \fint_{\mathbb{P}_1(0)} |\partial_{\mathbf{d}} \mathbf{f}(\mathbf{d}_i)|^3 dx \, dt \leqslant C \left(\fint_{\mathbb{P}_1(0)} |\mathbf{d}_i|^6 dx \, dt + 1 \right) \leqslant C(\varepsilon_i^6 + M_0^6 + 1). \end{cases}$$

$$\tag{4.9}$$

Furthermore, $(\widehat{\mathbf{u}}_i, \widehat{\mathbf{d}}_i, \widehat{P}_i)$ is a suitable weak solution of the blowing-up version of (1.2):

$$\begin{cases} \partial_t \widehat{\mathbf{u}}_i + \varepsilon_i \widehat{\mathbf{u}}_i \cdot \nabla \widehat{\mathbf{u}}_i + \nabla \widehat{P}_i \\[2mm] = \Delta \widehat{\mathbf{u}}_i - \varepsilon_i \nabla \widehat{\mathbf{d}}_i \cdot \Delta \widehat{\mathbf{d}}_i + \dfrac{r_i^2}{\varepsilon_i} \nabla \mathbf{d}_i \cdot \mathbf{f}(\mathbf{d}_i) - \nabla \cdot S_\alpha \left[\Delta \widehat{\mathbf{d}}_i - \dfrac{r_i^2}{\varepsilon_i} \mathbf{f}(\mathbf{d}_i), \mathbf{d}_i \right], \\[2mm] \operatorname{div} \widehat{\mathbf{u}}_i = 0, \\[2mm] \partial_t \widehat{\mathbf{d}}_i + \varepsilon_i \widehat{\mathbf{u}}_i \cdot \nabla \widehat{\mathbf{d}}_i - T_\alpha [\nabla \widehat{\mathbf{u}}_i, \mathbf{d}_i] = \Delta \widehat{\mathbf{d}}_i - \dfrac{r_i^2}{\varepsilon_i} \mathbf{f}(\mathbf{d}_i). \end{cases} \tag{4.10}$$

From (4.8), we assume that there exists

$$(\widehat{\mathbf{u}}, \widehat{\mathbf{d}}, \widehat{P}) \in L^3(\mathbb{P}_1(0)) \times L_t^3 W_x^{1,3}(\mathbb{P}_1(0)) \times L^{\frac{3}{2}}(\mathbb{P}_1(0)) \tag{4.11}$$

such that, after passing to a subsequence,

$$(\widehat{\mathbf{u}}_i, \widehat{\mathbf{d}}_i, \widehat{P}_i) \rightharpoonup (\widehat{\mathbf{u}}, \widehat{\mathbf{d}}, \widehat{P}) \quad \text{in } L^3(\mathbb{P}_1(0)) \times L_t^3 W_x^{1,3}(\mathbb{P}_1(0)) \times L^{\frac{3}{2}}(\mathbb{P}_1(0)).$$

It follows from (4.8) and the lower semicontinuity that

$$\int_{\mathbb{P}_1(0)} \left(|\widehat{\mathbf{u}}|^3 + |\nabla\widehat{\mathbf{d}}|^3 \right) dx\, dt + \left(\int_{\mathbb{P}_1(0)} |\widehat{P}|^{\frac{3}{2}} dx\, dt \right)^2 + \left(\fint_{\mathbb{P}_1(0)} |\widehat{\mathbf{d}}|^6 dx\, dt \right)^{\frac{1}{2}} \leqslant 1. \tag{4.12}$$

We claim that

$$\|\widehat{\mathbf{u}}_i\|_{L_t^\infty L_x^2 \cap L_t^2 H_x^1(\mathbb{P}_{\frac{1}{2}}(0))} + \left\|\widehat{\mathbf{d}}_i\right\|_{L_t^\infty H_x^1 \cap L_t^2 H_x^2(\mathbb{P}_{\frac{1}{2}}(0))} \leqslant C < \infty. \tag{4.13}$$

We choose a cut-off function $\phi \in C_0^\infty(\mathbb{P}_1(0))$ such that

$$0 \leqslant \phi \leqslant 1, \phi \equiv 1 \quad \text{on } \mathbb{P}_{\frac{1}{2}}(0), \quad \text{and} \quad |\partial_t\phi| + |\nabla\phi| + |\nabla^2\phi| \leqslant C.$$

Define

$$\phi_i(x, t) := \phi\left(\frac{x - x_i}{r_i}, \frac{t - t_i}{r_i^2}\right), \quad \forall\, (x, t) \in \mathbb{R}^3 \times (0, \infty).$$

Replacing ϕ by ϕ_i^2 in (1.7), by Young's inequality we can show

$$\begin{aligned}
\sup_{t_i - \frac{r_i^2}{4} \leqslant t \leqslant t_i} &\int_{B_{r_i}(x_i)} \left(|\mathbf{u}|^2 + |\nabla\mathbf{d}|^2 + F(\mathbf{d}) \right) \phi_i^2\, dx \\
&+ \int_{\mathbb{P}_{r_i}(z_i)} \left(|\nabla\mathbf{u}|^2 + |\Delta\mathbf{d}|^2 + |\mathbf{f}(\mathbf{d})|^2 \right) \phi_i^2\, dx\, dt \\
&\leqslant C \Bigg[\int_{\mathbb{P}_{r_i}(z_i)} (|\mathbf{u}|^2 + |\nabla\mathbf{d}|^2)|(\partial_t + \Delta)\phi_i^2| + F(\mathbf{d})|\partial_t\phi_i^2|dx\, dt \\
&\quad + \int_{\mathbb{P}_{r_i}(z_i)} (|\mathbf{u}|^2 + |\nabla\mathbf{d}|^2 + |P|)|\mathbf{u}||\nabla\phi_i^2|dx\, dt \\
&\quad + \int_{\mathbb{P}_{r_i}(z_i)} |\mathbf{d}|^2|\mathbf{u}|^2|\nabla\phi_i|^2 + |\mathbf{d}|^2|\nabla\mathbf{d}|^2|\nabla\phi_i|^2 dx\, dt \\
&\quad + \int_{\mathbb{P}_{r_i}(z_i)} |\nabla\mathbf{d}|^2(|\nabla^2(\phi_i^2)| + |\nabla\phi_i|^2) + |\partial_\mathbf{d}\mathbf{f}(\mathbf{d})||\nabla\mathbf{d}|^2\phi_i^2\, dx\, dt \Bigg].
\end{aligned} \tag{4.14}$$

By rescaling and using the estimates (4.7), (4.8), and (4.9), we can show that

$$
\sup_{-\frac{1}{4} \leqslant t \leqslant 0} \int_{B_{\frac{1}{2}}(0)} \left(|\widehat{\mathbf{u}}_i|^2 + |\nabla \widehat{\mathbf{d}}_i|^2 \right) \mathrm{d}x + \int_{\mathbb{P}_{\frac{1}{2}}(0)} \left(|\nabla \widehat{\mathbf{u}}_i|^2 + |\nabla^2 \widehat{\mathbf{d}}_i|^2 \right) \mathrm{d}x \, \mathrm{d}t
$$

$$
\leqslant C \int_{\mathbb{P}_1(0)} \left[\left(|\widehat{\mathbf{u}}_i|^2 + |\nabla \widehat{\mathbf{d}}_i|^2 \right) + \frac{r_i^2}{\varepsilon_i^2} F(\mathbf{d}_i) \right] \mathrm{d}x \, \mathrm{d}t
$$

$$
+ C \varepsilon_i \int_{\mathbb{P}_1(0)} (|\widehat{\mathbf{u}}_i|^2 + |\nabla \widehat{\mathbf{d}}_i|^2 + |\widehat{P}_i|) |\widehat{\mathbf{u}}_i| \mathrm{d}x \, \mathrm{d}t \tag{4.15}
$$

$$
+ C \int_{\mathbb{P}_1(0)} (|\mathbf{d}_i|^2 |\widehat{\mathbf{u}}_i|^2 + |\mathbf{d}_i|^2 |\nabla \widehat{\mathbf{d}}_i|^2) \, \mathrm{d}x \, \mathrm{d}t
$$

$$
+ C \int_{\mathbb{P}_1(0)} |\nabla \widehat{\mathbf{d}}_i|^2 + r_i^2 |\nabla \widehat{\mathbf{d}}_i|^2 |\partial_{\mathbf{d}} \mathbf{f}(\mathbf{d}_i)| \mathrm{d}x \, \mathrm{d}t
$$

$$
\leqslant C.
$$

This yields (4.13). Hence we may assume that

$$
(\widehat{\mathbf{u}}_i, \widehat{\mathbf{d}}_i) \rightharpoonup (\widehat{\mathbf{u}}, \widehat{\mathbf{d}}) \quad \text{in } L_t^2 H_x^1(\mathbb{P}_{\frac{1}{2}}(0)) \times L_t^2 H_x^2(\mathbb{P}_{\frac{1}{2}}(0)). \tag{4.16}
$$

From $\frac{r_i}{\varepsilon_i} \to 0$ and $|\fint_{\mathbb{P}_1(0)} \mathbf{d}_i \mathrm{d}x \, \mathrm{d}t| \leqslant M_0$, we have

$$
\left| \fint_{\mathbb{P}_{\frac{1}{2}}(0)} \mathbf{d}_i \, \mathrm{d}x \, \mathrm{d}t \right| \leqslant \left| \fint_{\mathbb{P}_{\frac{1}{2}}(0)} \left(\mathbf{d}_i - \fint_{\mathbb{P}_1(0)} \mathbf{d}_i \right) \mathrm{d}x \, \mathrm{d}t \right| + \left| \fint_{\mathbb{P}_1(0)} \mathbf{d}_i \mathrm{d}x \, \mathrm{d}t \right|
$$

$$
\leqslant C \left(\fint_{\mathbb{P}_1(0)} |\mathbf{d}_i - \overline{\mathbf{d}}_i|^6 \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{6}} + M_0 \leqslant C \varepsilon_i + M_0 \leqslant C.
$$

Thus by the same interpolation as in (3.11), we have

$$
\|\mathbf{d}_i\|_{L^{10}(\mathbb{P}_{\frac{1}{2}}(0))} \leqslant C,
$$

$$
\fint_{\mathbb{P}_{\frac{1}{2}}(0)} |\mathbf{f}(\mathbf{d}_i)|^{\frac{10}{3}} \mathrm{d}x \, \mathrm{d}t \leqslant C,
$$

$$
\fint_{\mathbb{P}_{\frac{1}{2}}(0)} |\mathbf{d}_i \otimes \mathbf{f}(\mathbf{d}_i)|^{\frac{5}{2}} \mathrm{d}x \, \mathrm{d}t \leqslant C
$$

$$
\fint_{\mathbb{P}_{\frac{1}{2}}(0)} F(\mathbf{d}_i)^{\frac{5}{2}} \mathrm{d}x \, \mathrm{d}t \leqslant C,
$$

and there exists a constant $\overline{\mathbf{d}} \in \mathbb{R}^3$, with $|\overline{\mathbf{d}}| \leqslant M_0$, such that, after passing to subsequence, $\overline{\mathbf{d}}_i \to \overline{\mathbf{d}}$,

$$
\mathbf{d}_i \to \overline{\mathbf{d}} \quad \text{in } L^6(\mathbb{P}_{\frac{1}{2}}(0)),
$$

and

$$\frac{r_i^2}{\varepsilon_i}\mathbf{f}(\mathbf{d}_i) \to 0 \quad \text{in } L^{\frac{10}{3}}(\mathbb{P}_{\frac{1}{2}}(0)),$$

$$\frac{r_i^2}{\varepsilon_i}\mathbf{f}(\mathbf{d}_i) \otimes \mathbf{d}_i \to 0 \quad \text{in } L^{\frac{5}{2}}(\mathbb{P}_{\frac{1}{2}}(0)),$$

$$\frac{r_i^2}{\varepsilon_i}\mathbf{d}_i \otimes \mathbf{f}(\mathbf{d}_i) \to 0 \quad \text{in } L^{\frac{5}{2}}(\mathbb{P}_{\frac{1}{2}}(0)),$$

$$\frac{r_i^2}{\varepsilon_i^2}F(\mathbf{d}_i) \to 0 \quad \text{in } L^{\frac{5}{2}}(\mathbb{P}_{\frac{1}{2}}(0)).$$

(4.17)

Hence $(\widehat{\mathbf{u}}, \widehat{\mathbf{d}}, \widehat{P}) : \mathbb{P}_{\frac{1}{2}}(0) \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$ solves the linear system:

$$\begin{cases} \partial_t\widehat{\mathbf{u}} + \nabla\widehat{P} - \Delta\widehat{\mathbf{u}} = -\nabla \cdot S_\alpha[\Delta\widehat{\mathbf{d}}, \overline{\mathbf{d}}], \\ \operatorname{div}\widehat{\mathbf{u}} = 0, \\ \partial_t\widehat{\mathbf{d}} - \Delta\widehat{\mathbf{d}} = T_\alpha[\nabla\widehat{\mathbf{u}}, \overline{\mathbf{d}}]. \end{cases}$$

(4.18)

By lemma 4.3 and (4.12), we have that $(\widehat{\mathbf{u}}, \widehat{\mathbf{d}}) \in C^\infty(\mathbb{P}_{\frac{1}{4}}), \widehat{P} \in L^\infty\left(-\left[\frac{1}{16}, 0\right], C^\infty(B_{\frac{1}{4}}(0))\right)$ satisfies

$$\tau^{-2}\int_{\mathbb{P}_\tau(0)}\left(|\widehat{\mathbf{u}}|^3 + |\nabla\widehat{\mathbf{d}}|^3\right)\mathrm{d}x\,\mathrm{d}t + \left(\tau^{-2}\int_{\mathbb{P}_\tau(0)}|\widehat{P}|^{\frac{3}{2}}\mathrm{d}x\,\mathrm{d}t\right)^2$$

$$\leqslant C\tau^3\left[\int_{\mathbb{P}_{\frac{1}{2}}(0)}\left(|\widehat{\mathbf{u}}|^3 + |\nabla\widehat{\mathbf{d}}|^3\right)\mathrm{d}x\,\mathrm{d}t + \left(\int_{\mathbb{P}_{\frac{1}{2}}(0)}|\widehat{P}|^{\frac{3}{2}}\mathrm{d}x\,\mathrm{d}t\right)^2\right]$$

(4.19)

$$\leqslant C\tau^3, \ \forall \ \tau \in \left(0, \frac{1}{8}\right).$$

and $\exists\alpha_0 \in (0, 1)$ such that

$$\left(\fint_{\mathbb{P}_\tau(0)}|\widehat{\mathbf{d}} - \widehat{\mathbf{d}}_{0,\tau}|^6\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} \leqslant C\left(\fint_{\mathbb{P}_{\frac{1}{2}}(0)}|\widehat{\mathbf{d}}|^6\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}\tau^{3\alpha_0} \leqslant C\tau^{3\alpha_0}, \ \forall \ \tau \in \left(0, \frac{1}{8}\right).$$

(4.20)

We now claim that

$$\begin{aligned} (\widehat{\mathbf{u}}_i, \nabla\widehat{\mathbf{d}}_i) &\to (\widehat{\mathbf{u}}, \nabla\widehat{\mathbf{d}}) &&\text{in } L^3(\mathbb{P}_{\frac{3}{8}}(0)), \\ \widehat{\mathbf{d}}_i &\to \widehat{\mathbf{d}} &&\text{in } L^6(\mathbb{P}_{\frac{3}{8}}(0)). \end{aligned}$$

(4.21)

In fact, from the equation for $\widehat{\mathbf{u}}_i$ and $\widehat{\mathbf{d}}_i$ in (4.10) we can conclude that

$$\|\partial_t \widehat{\mathbf{u}}_i\|_{L_t^2 H_x^{-1} + L_t^{\frac{6}{5}} L_x^{\frac{6}{5}} + L_t^{\frac{3}{2}} W_x^{-1,\frac{3}{2}} (\mathbb{P}_{\frac{3}{8}}(0))} \leqslant C,$$

and

$$\|\partial_t \widehat{\mathbf{d}}_i\|_{L^{\frac{3}{2}}(\mathbb{P}_{\frac{3}{8}}(0))} \leqslant C. \tag{4.22}$$

Thus (4.21) follows from Aubin–Lions' compactness Lemma. This implies that for any $\tau \in (0, \frac{1}{8})$,

$$\tau^{-2} \int_{\mathbb{P}_\tau(0)} \left(|\widehat{\mathbf{u}}_i|^3 + |\nabla\widehat{\mathbf{d}}_i|^3 \right) dx\, dt = \tau^{-2} \int_{\mathbb{P}_\tau(0)} (|\widehat{\mathbf{u}}|^3 + |\nabla\widehat{\mathbf{d}}|^3)\, dx\, dt + \tau^{-2} o(1)$$

$$\leqslant C\tau^3 + \tau^{-2} o(1),$$

$$\left(\fint_{\mathbb{P}_\tau(0)} |\widehat{\mathbf{d}}_i - (\widehat{\mathbf{d}}_i)_{0,\tau}|^6 dx\, dt \right)^{\frac{1}{2}} \leqslant C\tau^{3\alpha_0} + o(1), \tag{4.23}$$

where $\lim_{i\to\infty} o(1) = 0$.

Now we need to estimate the pressure $\widehat{P}_i$. By taking divergence of the $\widehat{\mathbf{u}}_i$ equation in (4.18) we see that

$$-\Delta\widehat{P}_i = \varepsilon_i \operatorname{div}^2 \left[\widehat{\mathbf{u}}_i \otimes \widehat{\mathbf{u}}_i + \nabla\widehat{\mathbf{d}}_i \odot \nabla\widehat{\mathbf{d}}_i - \left(\frac{1}{2}|\nabla\widehat{\mathbf{d}}_i|^2 + \frac{r_i^2}{\varepsilon_i^2} F(\mathbf{d}_i) \right) I_3 \right]$$

$$+ \operatorname{div}^2 S_\alpha \left[\Delta\widehat{\mathbf{d}}_i - \frac{r_i^2}{\varepsilon_i} \mathbf{f}(\mathbf{d}_i), \mathbf{d}_i \right] \quad \text{in } B_1. \tag{4.24}$$

We claim that

$$\tau^{-2} \int_{\mathbb{P}_\tau(0)} |\widehat{P}_i|^{\frac{3}{2}} dx\, dt \leqslant C\tau + C\tau^{-2}(\varepsilon_i + o(1)). \tag{4.25}$$

Since $S_\alpha[\Delta\widehat{\mathbf{d}}_i, \mathbf{d}_i]$ does not necessarily have a small L^2-norm in $\mathbb{P}_{\frac{1}{2}}(0)$, to achieve (4.25) we will show the following strong convergence in L^2:

$$(\nabla\widehat{\mathbf{u}}_i, \Delta\widehat{\mathbf{d}}_i) \to (\nabla\widehat{\mathbf{u}}, \Delta\widehat{\mathbf{d}}) \quad \text{in } L^2\left(\mathbb{P}_{\frac{3}{8}}(0) \right). \tag{4.26}$$

In order to prove (4.26), first observe that by subtracting the equation (4.10) from the equations (4.18), we see that

$$(\widetilde{\mathbf{u}}_i, \widetilde{\mathbf{d}}_i, \widetilde{P}_i) := \left(\widehat{\mathbf{u}}_i - \widehat{\mathbf{u}}, \widehat{\mathbf{d}}_i - \widehat{\mathbf{d}}, \widehat{P}_i - \widehat{P} \right)$$

solves the following system of equations in $\mathbb{P}_{\frac{1}{2}}(0)$:

$$
\begin{cases}
\partial_t \widetilde{\mathbf{u}}_i - \Delta \widetilde{\mathbf{u}}_i + \nabla \widetilde{P}_i = -\varepsilon_i \widehat{\mathbf{u}}_i \cdot \nabla \widehat{\mathbf{u}}_i - \varepsilon_i \nabla \widehat{\mathbf{d}}_i \cdot \Delta \widehat{\mathbf{d}}_i + \dfrac{r_i^2}{\varepsilon_i} \nabla \mathbf{d}_i \cdot \mathbf{f}(\mathbf{d}_i) \\[2mm]
\quad -\nabla \cdot S_\alpha \left[\Delta \widehat{\mathbf{d}}_i - \dfrac{r_i^2}{\varepsilon_i} \mathbf{f}(\mathbf{d}_i), \mathbf{d}_i \right] + \nabla \cdot S_\alpha [\Delta \widehat{\mathbf{d}}, \overline{\mathbf{d}}], \\[2mm]
\operatorname{div} \widetilde{\mathbf{u}}_i = 0, \\[2mm]
\partial_t \widetilde{\mathbf{d}}_i - \Delta \widetilde{\mathbf{d}}_i = -\varepsilon_i \widehat{\mathbf{u}}_i \cdot \nabla \widehat{\mathbf{d}}_i - \dfrac{r_i^2}{\varepsilon_i} \mathbf{f}(\mathbf{d}_i) + T_\alpha[\nabla \widehat{\mathbf{u}}_i, \mathbf{d}_i] - T_\alpha[\nabla \widehat{\mathbf{u}}, \overline{\mathbf{d}}].
\end{cases}
\tag{4.27}
$$

Since $(\widehat{\mathbf{u}}_i, \widehat{\mathbf{d}}_i, \widehat{P}_i)$ is a suitable weak solution of (4.10) and lemma 4.3 guarantees the smoothness of $(\widehat{\mathbf{u}}, \widehat{\mathbf{d}}, \widehat{P})$, it is not hard to see that (4.27) also enjoys a local energy inequality which leads to (4.26). In fact, multiplying the $\widetilde{\mathbf{u}}_i$ equation by $\widetilde{\mathbf{u}}_i \phi$, and $\nabla \widetilde{\mathbf{d}}_i$ equation by $\nabla \widetilde{\mathbf{d}}_i \phi$, integrating the resulting equation over $\mathbb{R}^3 \times [0, T]$, and applying the integration by parts, we obtain that

$$
\int_{\mathbb{R}^3} |\widetilde{\mathbf{u}}_i|^2 \phi(x, t) \mathrm{d}x + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla \widetilde{\mathbf{u}}_i|^2 \phi \, \mathrm{d}x \, \mathrm{d}s
$$

$$
\leqslant \int_0^t \int_{\mathbb{R}^3} |\widetilde{\mathbf{u}}_i|^2 (\partial_t \phi + \Delta \phi) \, \mathrm{d}x \, \mathrm{d}s
$$

$$
+ \int_0^t \int_{\mathbb{R}^3} [\varepsilon_i |\widehat{\mathbf{u}}_i|^2 \widehat{\mathbf{u}}_i \cdot \nabla \phi + 2\varepsilon_i (\widehat{\mathbf{u}}_i \cdot \nabla \widehat{\mathbf{u}}_i) \cdot \widehat{\mathbf{u}} \phi + 2 \widetilde{P}_i \widetilde{\mathbf{u}}_i \cdot \nabla \phi] \, \mathrm{d}x \, \mathrm{d}s
$$

$$
+ 2 \int_0^t \int_{\mathbb{R}^3} \left(-\varepsilon_i \nabla \widehat{\mathbf{d}}_i \cdot \Delta \widehat{\mathbf{d}}_i \cdot (\widehat{\mathbf{u}}_i - \widehat{\mathbf{u}}) \phi + \dfrac{r_i^2}{\varepsilon_i} \nabla \mathbf{d}_i \cdot \mathbf{f}(\mathbf{d}_i) \cdot \widetilde{\mathbf{u}}_i \phi \right) \mathrm{d}x \, \mathrm{d}s
$$

$$
- \dfrac{2 r_i^2}{\varepsilon_i} \int_0^t \int_{\mathbb{R}^3} S_\alpha[\mathbf{f}(\mathbf{d}_i), \mathbf{d}_i] : (\nabla \widetilde{\mathbf{u}}_i \phi + \widetilde{\mathbf{u}}_i \otimes \nabla \phi) \, \mathrm{d}x \, \mathrm{d}s
$$

$$
+ 2 \int_0^t \int_{\mathbb{R}^3} \left(S_\alpha[\Delta \widehat{\mathbf{d}}_i, \mathbf{d}_i] - S_\alpha[\Delta \widehat{\mathbf{d}}, \overline{\mathbf{d}}] \right) : \left(\nabla \widehat{\mathbf{u}}_i \phi - \nabla \widehat{\mathbf{u}} \phi + \widetilde{\mathbf{u}}_i \otimes \nabla \phi \right) \mathrm{d}x \mathrm{d}s,
\tag{4.28}
$$

and

$$
\int_{\mathbb{R}^3} |\nabla \widehat{\mathbf{d}}_i|^2 \phi(x, t) \mathrm{d}x + 2 \int_0^t \int_{\mathbb{R}^3} |\Delta \widetilde{\mathbf{d}}_i|^2 \phi \, \mathrm{d}x \, \mathrm{d}s
$$

$$
\leqslant \int_0^t \int_{\mathbb{R}^3} (|\nabla \widetilde{\mathbf{d}}_i|^2 (\partial_t \phi + \Delta \phi) + 2\varepsilon_i \widehat{\mathbf{u}}_i \cdot \nabla \widehat{\mathbf{d}}_i \cdot (\Delta \widehat{\mathbf{d}}_i \phi - \Delta \widehat{\mathbf{d}} \phi + \nabla \phi \cdot \nabla \widetilde{\mathbf{d}}_i)) \, \mathrm{d}x \, \mathrm{d}s
$$

$$
+ \dfrac{2 r_i^2}{\varepsilon_i} \int_0^t \int_{\mathbb{R}^3} \mathbf{f}(\mathbf{d}_i) \cdot (\Delta \widetilde{\mathbf{d}}_i \phi + \nabla \phi \cdot \nabla \widetilde{\mathbf{d}}_i) \, \mathrm{d}x \, \mathrm{d}s
$$

$$
- 2 \int_0^t \int_{\mathbb{R}^3} \left[T_\alpha[\nabla \widehat{\mathbf{u}}_i, \mathbf{d}_i] - T_\alpha[\nabla \widehat{\mathbf{u}}, \overline{\mathbf{d}}] \right] \cdot \left(\Delta \widehat{\mathbf{d}}_i \phi - \Delta \widehat{\mathbf{d}} \phi + \nabla \phi \cdot \nabla \widetilde{\mathbf{d}}_i \right) \mathrm{d}x \, \mathrm{d}s. \tag{4.29}
$$

Recall that

$$
\int_0^t \int_{\mathbb{R}^3} S_\alpha[\Delta \widehat{\mathbf{d}}_i, \mathbf{d}_i] : \nabla \widehat{\mathbf{u}}_i \phi \mathrm{d}x \, \mathrm{d}t = \int_0^t \int_{\mathbb{R}^3} T_\alpha[\nabla \widehat{\mathbf{u}}_i, \mathbf{d}_i] \cdot \Delta \widehat{\mathbf{d}}_i \phi \, \mathrm{d}x \, \mathrm{d}t, \tag{4.30}
$$

$$
\int_0^t \int_{\mathbb{R}^3} S_\alpha[\Delta \widehat{\mathbf{d}}, \overline{\mathbf{d}}] : \nabla \widehat{\mathbf{u}} \phi \, \mathrm{d}x \, \mathrm{d}t = \int_0^t \int_{\mathbb{R}^3} T_\alpha[\nabla \mathbf{u}, \overline{\mathbf{d}}] : \Delta \widehat{\mathbf{d}} \phi \, \mathrm{d} \, \mathrm{d}t. \tag{4.31}
$$

Therefore we can add (4.28) and (4.29) to obtain that

$$
\int_{\mathbb{R}^3}(|\widetilde{\mathbf{u}}_i|^2 + |\nabla\widetilde{\mathbf{d}}_i|^2)\phi(x,t)\mathrm{d}x + 2\int_0^t\int_{\mathbb{R}^3}\left(|\nabla\widetilde{\mathbf{u}}_i|^2 + |\Delta\widetilde{\mathbf{d}}_i|^2\right)\phi\,\mathrm{d}x\,\mathrm{d}s
$$

$$
\begin{aligned}
&\leqslant \int_0^t\int_{\mathbb{R}^3}\left[(|\widetilde{\mathbf{u}}_i|^2 + |\nabla\widetilde{\mathbf{d}}_i|^2)(\partial_t\phi + \Delta\phi)\right.\\
&\qquad\left. + (\varepsilon_i|\widehat{\mathbf{u}}_i|^2\widehat{\mathbf{u}}_i + 2\widetilde{P}_i\widetilde{\mathbf{u}}_i)\cdot\nabla\phi + 2\varepsilon_i(\widehat{\mathbf{u}}_i\cdot\nabla\widehat{\mathbf{u}}_i)\cdot\widehat{\mathbf{u}}\phi\right]\mathrm{d}x\,\mathrm{d}s\\
&\quad + 2\int_0^t\int_{\mathbb{R}^3}\varepsilon_i\nabla\widehat{\mathbf{d}}_i\cdot\Delta\widehat{\mathbf{d}}_i\cdot\widehat{\mathbf{u}}\phi - \varepsilon_i\widehat{\mathbf{u}}_i\cdot\nabla\widehat{\mathbf{d}}_i\cdot\Delta\widehat{\mathbf{d}}\phi + 2\varepsilon_i\widehat{\mathbf{u}}_i\cdot\nabla\widehat{\mathbf{d}}_i\cdot(\nabla\phi\cdot\nabla\widetilde{\mathbf{d}}_i)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad + \frac{2r_i^2}{\varepsilon_i}\int_0^t\int_{\mathbb{R}^3}\nabla\mathbf{d}_i\cdot\mathbf{f}(\mathbf{d}_i)\cdot\widetilde{\mathbf{u}}_i\phi + \mathbf{f}(\mathbf{d}_i)\cdot\left(\Delta\widetilde{\mathbf{d}}_i\phi + \nabla\phi\cdot\nabla\widetilde{\mathbf{d}}_i\right)\mathrm{d}x\,\mathrm{d}s\\
&\quad - \frac{2r_i^2}{\varepsilon_i^2}\int_0^t\int_{\mathbb{R}^3}S_\alpha[\mathbf{f}(\mathbf{d}_i),\mathbf{d}_i]:(\nabla\widetilde{\mathbf{u}}_i\phi + \widetilde{\mathbf{u}}_i\otimes\nabla\phi)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad + 2\int_0^t\int_{\mathbb{R}^3}S_\alpha[\Delta\widehat{\mathbf{d}}_i,\mathbf{d}_i]:(\widetilde{\mathbf{u}}_i\otimes\nabla\phi - \nabla\widehat{\mathbf{u}}\phi)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad - 2\int_0^t\int_{\mathbb{R}^3}T_\alpha[\nabla\widehat{\mathbf{u}}_i,\mathbf{d}_i]\cdot(\nabla\phi\cdot\nabla\widetilde{\mathbf{d}}_i - \Delta\widehat{\mathbf{d}}\phi)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad - 2\int_0^t\int_{\mathbb{R}^3}S_\alpha[\Delta\widehat{\mathbf{d}},\overline{\mathbf{d}}]:(\nabla\widehat{\mathbf{u}}_i\phi + \widetilde{\mathbf{u}}_i\otimes\nabla\phi)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad + 2\int_0^t\int_{\mathbb{R}^3}T_\alpha[\nabla\widehat{\mathbf{u}},\overline{\mathbf{d}}]\cdot(\Delta\widehat{\mathbf{d}}_i\phi + \nabla\phi\cdot\nabla\widetilde{\mathbf{d}}_i)\,\mathrm{d}x\,\mathrm{d}s\\
&:= \sum_{k=1}^{8}I_k(i).
\end{aligned}
\tag{4.32}
$$

From the convergence (4.16), we know that

$$
\lim_{i\to\infty}\left\|(\widetilde{\mathbf{u}}_i,\nabla\widetilde{\mathbf{d}}_i)\right\|_{L^3(\mathbb{P}_{\frac{3}{8}}(0))} = 0,
$$

$$
\widetilde{P}_i \rightharpoonup 0 \quad\text{in } L^{\frac{3}{2}}(\mathbb{P}_{\frac{3}{8}}(0)),
$$

$$
(\nabla\widetilde{\mathbf{u}}_i,\nabla^2\widetilde{\mathbf{d}}_i) \rightharpoonup (0,0) \quad\text{in } L^2(\mathbb{P}_{\frac{3}{8}}(0)).
$$

This, together with (4.17), implies that as $i\to\infty$, $\sum_{k=1}^4 I_k\to 0$ and

$$
\begin{aligned}
I_5 &\to -2\int_0^t\int_{\mathbb{R}^3}S_\alpha[\Delta\widehat{\mathbf{d}},\overline{\mathbf{d}}]:\nabla\widehat{\mathbf{u}}\phi\,\mathrm{d}x\,\mathrm{d}s\\
I_6 &\to 2\int_0^t\int_{\mathbb{R}^3}T_\alpha[\nabla\widehat{\mathbf{u}},\overline{\mathbf{d}}]\cdot\Delta\widehat{\mathbf{d}}\phi\mathrm{d}x\,\mathrm{d}s\\
I_7 &\to -2\int_0^t\int_{\mathbb{R}^3}S_\alpha[\Delta\widehat{\mathbf{d}},\overline{\mathbf{d}}]:\nabla\widehat{\mathbf{u}}\phi\,\mathrm{d}x\,\mathrm{d}s\\
I_8 &\to 2\int_0^t\int_{\mathbb{R}^3}T_\alpha[\nabla\widehat{\mathbf{u}},\overline{\mathbf{d}}]:\Delta\widehat{\mathbf{d}}\phi\,\mathrm{d}x\,\mathrm{d}s,
\end{aligned}
\tag{4.33}
$$

Therefore

$$\sum_{k=1}^{8} I_k(i) \to 4 \int_0^t \int_{\mathbb{R}^3} T_\alpha[\nabla \widehat{\mathbf{u}}, \overline{\mathbf{d}}] : \Delta \widehat{\mathbf{d}} \phi - S_\alpha[\Delta \widehat{\mathbf{d}}, \overline{\mathbf{d}}] : \nabla \widehat{\mathbf{u}} \phi \, dx \, ds = 0,$$

and (4.26) holds.

Let $\eta \in C_0^\infty(B_{\frac{3}{8}}(0))$ be such that $\eta \equiv 1$ in $B_{\frac{5}{16}}(0), 0 \leqslant \eta \leqslant 1$. For any $-(\frac{3}{8})^2 \leqslant t \leqslant 0$, define $\widehat{P}_i^{(1)}(\cdot, t) : \mathbb{R}^3 \to \mathbb{R}$ by

$$\widehat{P}_i^{(1)}(x, t) = \int_{\mathbb{R}^3} \nabla_x^2 G(x - y) \left\{ \varepsilon_i \eta \left[\widehat{\mathbf{u}}_i \otimes \widehat{\mathbf{u}}_i + \nabla \widehat{\mathbf{d}}_i \odot \nabla \widehat{\mathbf{d}}_i \right. \right.$$
$$\left. - \left(\frac{1}{2} |\nabla \widehat{\mathbf{d}}_i|^2 + \frac{r_i^2}{\varepsilon_i^2} F(\mathbf{d}_i) \right) I_3 \right] - \frac{r_i^2}{\varepsilon_i} [S_\alpha[\mathbf{f}(\mathbf{d}_i), \mathbf{d}_i]]$$
$$\left. + \left[S_\alpha[\Delta \widehat{\mathbf{d}}_i, \mathbf{d}_i] - S_\alpha[\Delta \widehat{\mathbf{d}}, \overline{\mathbf{d}}] \right] \right\} (y, t) dy, \tag{4.34}$$

and $\widehat{P}_i^{(2)}(\cdot, t) = (\widehat{P}_i - \widehat{P}_i^{(1)})(\cdot, t)$. Then

$$-\Delta \widehat{P}_i^{(2)} = \text{div}^2 \, S_\alpha[\Delta \widehat{\mathbf{d}}, \overline{\mathbf{d}}] \quad \text{in } B_{\frac{5}{16}}(0). \tag{4.35}$$

For $\widehat{P}_i^{(1)}$, by the Calderon–Zgymund theory we have that

$$\|\widehat{P}_i^{(1)}\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \leqslant C \left[\varepsilon_i \left(\|\widehat{\mathbf{u}}_i\|_{L^3(B_{\frac{3}{8}}(0))}^2 + \|\nabla \widehat{\mathbf{d}}_i\|_{L^3(B_{\frac{3}{8}}(0))}^2 + \frac{r_i^2}{\varepsilon_i^2} \|F(\mathbf{d}_i)\|_{L^{\frac{3}{2}}(B_{\frac{3}{8}}(0))} \right) \right.$$
$$\left. + \frac{r_i^2}{\varepsilon_i} \||\mathbf{f}(\mathbf{d}_i)||\mathbf{d}_i|\|_{L^{\frac{3}{2}}(B_{\frac{3}{8}}(0))} + \|S_\alpha[\Delta \widehat{\mathbf{d}}_i, \mathbf{d}_i] - S_\alpha[\Delta \widehat{\mathbf{d}}, \overline{\mathbf{d}}]\|_{L^{\frac{3}{2}}(B_{\frac{3}{8}}(0))} \right]$$
$$\leqslant C \left[\varepsilon_i \left(\|\widehat{\mathbf{u}}_i\|_{L^3(B_{\frac{3}{8}}(0))}^2 + \|\nabla \widehat{\mathbf{d}}_i\|_{L^3(B_{\frac{3}{8}}(0))}^2 + \frac{r_i^2}{\varepsilon_i^2} \|F(\mathbf{d}_i)\|_{L^{\frac{3}{2}}(B_{\frac{3}{8}}(0))} \right) \right.$$
$$+ \frac{r_i^2}{\varepsilon_i} \||\mathbf{f}(\mathbf{d}_i)||\mathbf{d}_i|\|_{L^{\frac{3}{2}}(B_{\frac{3}{8}}(0))} + \|\mathbf{d}_i\|_{L^6(B_{\frac{3}{8}}(0))} \|\Delta \widehat{\mathbf{d}}_i - \Delta \widehat{\mathbf{d}}\|_{L^2(B_{\frac{3}{8}}(0))}$$
$$\left. + \|\mathbf{d}_i - \overline{\mathbf{d}}\|_{L^6(B_{\frac{3}{8}}(0))} \|\Delta \widehat{\mathbf{d}}\|_{L^2(B_{\frac{3}{8}}(0))} \right] \tag{4.36}$$

Hence we have

$$\|\widehat{P}_i^{(1)}\|_{L^{\frac{3}{2}}(\mathbb{P}_{\frac{1}{3}}(0))} \leqslant C(\varepsilon_i + o(1)). \tag{4.37}$$

From the standard theory on linear elliptic equations, $\widehat{P}_i^{(2)} \in C^\infty(B_{\frac{5}{16}}(0))$ satisfies that for any $0 < \tau < \frac{9}{32}$,

$$
\tau^{-2} \int_{\mathbb{P}_\tau(0)} |\widehat{P}_i^{(2)}|^{\frac{3}{2}} \mathrm{d}x \leqslant C\tau \left[\int_{\mathbb{P}_{\frac{9}{32}}(0)} |\widehat{P}_i^{(2)}|^{\frac{3}{2}} \mathrm{d}x \, \mathrm{d}t + \|\nabla^3 \widehat{\mathbf{d}}\|_{L^\infty(\mathbb{P}_{\frac{9}{32}}(0))}^{\frac{3}{2}} \right]
$$

$$
\leqslant C\tau \left[\int_{\mathbb{P}_{\frac{9}{32}}(0)} (|\widehat{P}_i|^{\frac{3}{2}} + |\widehat{P}_i^{(1)}|^{\frac{3}{2}}) \, \mathrm{d}x \, \mathrm{d}t + \|\nabla^3 \widehat{\mathbf{d}}\|_{L^\infty(\mathbb{P}_{\frac{9}{32}}(0))}^{\frac{3}{2}} \right] \tag{4.38}
$$

$$
\leqslant C\tau(1 + \varepsilon_i + o(1)).
$$

Combining (4.37) with (4.38) yields (4.25). It follows from (4.23) and (4.25) that there exist sufficiently small $\tau_0 \in (0, \frac{1}{4})$ and sufficiently large i_0, depending on τ_0, such that for any $i \geqslant i_0$, it holds that

$$
\tau_0^{-2} \int_{\mathbb{P}_{\tau_0}(0)} (|\widehat{\mathbf{u}}_i|^3 + |\nabla \widehat{\mathbf{d}}_i|^3) \, \mathrm{d}x \, \mathrm{d}t + \left(\tau_0^{-2} \int_{\mathbb{P}_{\tau_0}(0)} |\widehat{P}_i|^{\frac{3}{2}} \mathrm{d}x \, \mathrm{d}t \right)^{\frac{3}{2}}
$$

$$
+ \left(\fint_{\mathbb{P}_{\tau_0}(0)} |\widehat{\mathbf{d}}_i - (\widehat{\mathbf{d}}_i)_{\tau_0,0}|^6 \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{2}} \leqslant \frac{1}{4}.
$$

This contradicts (4.8). The proof of lemma 4.1 is completed. $\qquad\square$

Now we will establish the smoothness of the limit equation (4.18) in the following lemma.

Lemma 4.3. *Assume that* $(\widehat{\mathbf{u}}, \widehat{\mathbf{d}}) \in (L_t^\infty L_x^2 \cap L_t^2 H_x^1)(\mathbb{P}_{\frac{1}{2}}(0)) \times \left(L_t^\infty H_x^1 \cap L_t^2 H_x^2 \right)(\mathbb{P}_{\frac{1}{2}})$ *and* $\widehat{P} \in L^{\frac{3}{2}}(\mathbb{P}_{\frac{1}{2}}(0))$ *is a weak solution of the linear system* (4.18)*, then* $(\widehat{\mathbf{u}}, \widehat{\mathbf{d}}) \in C^\infty(\mathbb{P}_{\frac{1}{4}}(0))$*, and the following estimate*

$$
\tau^{-2} \int_{\mathbb{P}_\tau(0)} \left(|\widehat{\mathbf{u}}|^3 + |\nabla \widehat{\mathbf{d}}|^3 + |\widehat{P}|^{\frac{3}{2}} \right) \mathrm{d}x \, \mathrm{d}t \leqslant C\tau^3 \int_{\mathbb{P}_{\frac{1}{2}}(0)} \left(|\widehat{\mathbf{u}}|^3 + |\nabla \widehat{\mathbf{d}}|^3 + |\widehat{P}|^{\frac{3}{2}} \right) \mathrm{d}x \, \mathrm{d}t
$$

$$\tag{4.39}$$

holds for any $\tau \in (0, \frac{1}{8})$.

Proof. The smoothness of the limit equation (4.18) doesn't follow from the standard theory of linear equations, since the source term of $\widehat{\mathbf{u}}$ equations involve terms depending on the third order derivatives of $\widehat{\mathbf{d}}$. It is based on higher order energy methods, for which the cancellation property, as in the derivation of local energy inequality for suitable weak solution to (1.2), plays a critical role. This strategy has been adapted by Huang–Lin–Wang in [11, lemma 3.2] for the full Ericksen–Leslie system in 2D. However, it is more delicate here due to the low temporal integrability of pressure. To address this issue, we split the pressure into two parts $\widehat{P}^{(1)}$ and $\widehat{P}^{(2)}$, where $\widehat{P}^{(1)}$ solves the Poisson equation involving $\Delta \widehat{\mathbf{d}}$ which belongs to L^2, and $\widehat{P}^{(2)}$, while is only $L^{\frac{3}{2}}$ in time, is harmonic in space. In fact, if we take the divergence of the equation (4.18)$_1$, then we have $\widehat{P}$ satisfies the following Poisson equation:

$$
-\Delta \widehat{P} = \mathrm{div}^2 \, S_\alpha[\Delta \widehat{\mathbf{d}}, \overline{\mathbf{d}}] \quad \text{in } \mathbb{P}_{\frac{1}{2}}. \tag{4.40}
$$

Now let $\zeta \in C_0^\infty(B_{\frac{1}{2}}(0))$ be a cut-off function of $B_{\frac{3}{8}}(0)$, i.e., $\zeta \equiv 1$ on $B_{\frac{3}{8}}(0), 0 \leqslant \zeta \leqslant 1$. Define $\widehat{P}^{(1)}(\cdot, t) : \mathbb{R}^3 \to \mathbb{R}$,

$$\widehat{P}^{(1)}(x, t) := \int_{\mathbb{R}^3} \nabla_x^2 G(x - y)\zeta(y)S_\alpha[\Delta\widehat{\mathbf{d}}, \overline{\mathbf{d}}](y, t)\mathrm{d}y,$$

and $\widehat{P}^{(2)}(\cdot, t) := (\widehat{P} - \widehat{P}^{(1)})(\cdot, t)$. For $\widehat{P}^{(1)}$, by Calderon–Zygmund's singular integral estimate we have

$$\left\|\widehat{P}^{(1)}(\cdot, t)\right\|_{L^2(\mathbb{R}^3)} \leqslant C\left\|\Delta\widehat{\mathbf{d}}(\cdot, t)\right\|_{L^2\left(B_{\frac{1}{2}}\right)}, \quad -\frac{1}{4} \leqslant t \leqslant 0.$$

Hence we can integrate the inequality above in time to get

$$\int_{\mathbb{P}_{\frac{1}{2}}} |\widehat{P}^{(1)}|^2 \mathrm{d}x \, \mathrm{d}t \leqslant C\int_{\mathbb{P}_{\frac{1}{2}}} |\Delta\widehat{\mathbf{d}}|^2 \mathrm{d}x \, \mathrm{d}t. \tag{4.41}$$

For $\widehat{P}^{(2)}$, it is easy to see that

$$-\Delta\widehat{P}^{(2)} = 0 \quad \text{in } B_{\frac{3}{8}}. \tag{4.42}$$

By the standard regularity theory of harmonic function we have

$$\begin{aligned}
\int_{\mathbb{P}_{\frac{5}{16}}} |\nabla^l \widehat{P}^{(2)}|^{\frac{3}{2}} \mathrm{d}x \, \mathrm{d}t &\leqslant C\int_{\mathbb{P}_{\frac{3}{8}}} |\widehat{P}^{(2)}|^{\frac{3}{2}} \mathrm{d}x \, \mathrm{d}t \\
&\leqslant C\int_{\mathbb{P}_{\frac{3}{8}}} (|\widehat{P}|^{\frac{3}{2}} + |\widehat{P}^{(1)}|^{\frac{3}{2}}) \, \mathrm{d}x \, \mathrm{d}t \\
&\leqslant C\int_{\mathbb{P}_{\frac{1}{2}}} |\widehat{P}|^{\frac{3}{2}} \mathrm{d}x \, \mathrm{d}t + C\int_{\mathbb{P}_{\frac{1}{2}}} |\widehat{P}^{(1)}|^2 \mathrm{d}x \, \mathrm{d}t + C \\
&\leqslant C\int_{\mathbb{P}_{\frac{1}{2}}} |\widehat{P}|^{\frac{3}{2}} \mathrm{d}x \, \mathrm{d}t + C\int_{\mathbb{P}_{\frac{1}{2}}} |\Delta\widehat{\mathbf{d}}|^2 \mathrm{d}x \, \mathrm{d}t + C, \quad l = 1, 2.
\end{aligned} \tag{4.43}$$

Taking $\frac{\partial}{\partial x_i}$ of the linear equation (4.18) yields

$$\begin{cases}
\partial_t \widehat{\mathbf{u}}_{x_i} + \nabla\widehat{P}_{x_i} - \Delta\widehat{\mathbf{u}}_{x_i} = -\nabla \cdot S_\alpha[\Delta\widehat{\mathbf{d}}, \overline{\mathbf{d}}]_{x_i}, \\
\nabla \cdot \widehat{\mathbf{u}}_{x_i} = 0, \\
\partial\widehat{\mathbf{d}}_{x_i} - \Delta\widehat{\mathbf{d}}_{x_i} = T_\alpha[\nabla\widehat{\mathbf{u}}, \overline{\mathbf{d}}]_{x_i}.
\end{cases} \tag{4.44}$$

For any $\eta \in C_0^\infty(B_{\frac{5}{16}})$, multiplying the equation (4.44)$_1$ by $\widehat{\mathbf{u}}_{x_i}\eta^2$ and the $\nabla\widehat{\mathbf{d}}_{x_i}$ equation from (4.44)$_3$ by $\nabla\widehat{\mathbf{d}}_{x_i}\eta^2$ and integrating the resulting equations over $B_{\frac{5}{16}}$, we obtain[1]

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{\frac{5}{16}}}|\nabla\widehat{\mathbf{u}}|^2\eta^2\mathrm{d}x + 2\int_{B_{\frac{5}{16}}}|\nabla^2\widehat{\mathbf{u}}|^2\eta^2\mathrm{d}x$$

$$= 2\int_{B_{\frac{5}{16}}}[\widehat{P}_{x_i}\widehat{\mathbf{u}}_{x_i}\cdot\nabla(\eta^2) - \nabla\widehat{\mathbf{u}}_{x_i}:\widehat{\mathbf{u}}_{x_i}\otimes\nabla(\eta^2)]\mathrm{d}x \tag{4.45}$$

$$+ 2\int_{B_{\frac{5}{16}}}[S_\alpha[\Delta\widehat{\mathbf{d}},\overline{\mathbf{d}}]_{x_i}:\widehat{\mathbf{u}}_{x_i}\otimes\nabla(\eta^2) + S_\alpha[\Delta\widehat{\mathbf{d}},\overline{\mathbf{d}}]_{x_i}:\nabla\widehat{\mathbf{u}}_{x_i}\eta^2]\mathrm{d}x.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{\frac{5}{16}}}|\nabla^2\widehat{\mathbf{d}}|^2\eta^2\mathrm{d}x + 2\int_{B_{\frac{5}{16}}}|\Delta\nabla\widehat{\mathbf{d}}|^2\eta^2\,\mathrm{d}x$$

$$= -2\int_{B_{\frac{5}{16}}}\nabla_j\nabla\widehat{\mathbf{d}}_{x_i}:\nabla\widehat{\mathbf{d}}_{x_i}\otimes\nabla_j(\eta^2)\mathrm{d}x \tag{4.46}$$

$$- 2\int_{B_{\frac{5}{16}}}T_\alpha[\nabla\widehat{\mathbf{u}},\overline{\mathbf{d}}]_{x_i}\cdot\nabla_j\widehat{\mathbf{d}}_{x_i}\nabla_j(\eta^2) + T_\alpha[\nabla\widehat{\mathbf{u}},\overline{\mathbf{d}}]_{x_i}\cdot\Delta\widehat{\mathbf{d}}_{x_i}\eta^2\,\mathrm{d}x.$$

Once again, we have the cancellation

$$\int_{B_{\frac{5}{16}}}[S_\alpha[\Delta\widehat{\mathbf{d}},\overline{\mathbf{d}}]_{x_i}:\nabla\widehat{\mathbf{u}}_{x_i}\eta^2 - T_\alpha[\nabla\widehat{\mathbf{u}},\overline{\mathbf{d}}]_{x_i}\cdot\Delta\widehat{\mathbf{d}}_{x_i}\eta^2]\mathrm{d}x$$

$$= \int_{B_{\frac{5}{16}}}[S_\alpha[\Delta\widehat{\mathbf{d}}_{x_i},\overline{\mathbf{d}}]:\nabla\widehat{\mathbf{u}}_{x_i}\eta^2 - T_\alpha[\nabla\widehat{\mathbf{u}}_{x_i},\overline{\mathbf{d}}]\cdot\Delta\widehat{\mathbf{d}}_{x_i}\eta^2]\mathrm{d}x = 0.$$

Now we add (4.45) and (4.46) together to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{\frac{5}{16}}}\left(|\nabla\widehat{\mathbf{u}}|^2 + |\nabla^2\widehat{\mathbf{d}}|^2\right)\eta^2\,\mathrm{d}x + \int_{B_{\frac{5}{16}}}\left(|\nabla^2\widehat{\mathbf{u}}|^2 + |\Delta\nabla\widehat{\mathbf{d}}|^2\right)\eta^2\,\mathrm{d}x$$

$$= 2\int_{B_{\frac{5}{16}}}\widehat{P}_{x_i}\widehat{\mathbf{u}}_{x_i}\cdot\nabla(\eta^2)\mathrm{d}x$$

$$- 2\int_{B_{\frac{5}{16}}}[\nabla\widehat{\mathbf{u}}_{x_i}:\widehat{\mathbf{u}}_{x_i}\otimes\nabla(\eta^2) + \nabla_j\nabla\widehat{\mathbf{d}}_{x_i}:\nabla\widehat{\mathbf{d}}_{x_i}\otimes\nabla_j(\eta^2)]\mathrm{d}x$$

$$+ 2\int_{B_{\frac{5}{16}}}[S_\alpha[\Delta\widehat{\mathbf{d}}_{x_i},\overline{\mathbf{d}}]:\widehat{\mathbf{u}}_{x_i}\otimes\nabla(\eta^2) - T_\alpha[\nabla\widehat{\mathbf{u}}_{x_i},\overline{\mathbf{d}}]\cdot\nabla_j\widehat{\mathbf{d}}_{x_i}\nabla_j(\eta^2)]\mathrm{d}x$$

$$:= I_1 + I_2 + I_3. \tag{4.47}$$

---

[1] Strictly speaking, we need to take finite quotient D_h^j of (4.18) ($j = 1, 2, 3$) and then sending $h \to 0$.

We have the following estimates:

$$
\begin{aligned}
|I_1| &\leqslant 2 \left| \int_{B_{\frac{5}{16}}} (\widehat{P}^{(1)} \widehat{\mathbf{u}}_{x_i x_i} \cdot \nabla(\eta^2) + \widehat{P}^{(1)} \widehat{\mathbf{u}}_{x_i} \cdot \nabla(\eta^2)_{x_i}) \mathrm{d}x \right| + 2 \left| \int_{B_{\frac{5}{16}}} \widehat{\mathbf{u}} \cdot (\widehat{P}^{(2)}_{x_i} \nabla(\eta^2))_{x_i} \mathrm{d}x \right| \\
&\leqslant \frac{1}{32} \int_{B_{\frac{5}{16}}} |\nabla^2 \widehat{\mathbf{u}}|^2 \eta^2 \mathrm{d}x + C \int_{B_{\frac{1}{2}}} (|\nabla \widehat{\mathbf{u}}|^2 \eta^2 + |\nabla \widehat{\mathbf{u}}|^2 |\nabla \eta|^2) \mathrm{d}x \\
&\quad + C \int_{\mathrm{spt}\,\eta} |\widehat{P}^{(1)}|^2 \mathrm{d}x + C \int_{\mathrm{spt}\,\eta} (|\widehat{\mathbf{u}}|^3 + |\nabla \widehat{P}^{(2)}|^{\frac{3}{2}} + |\nabla^2 \widehat{P}^{(2)}|^{\frac{3}{2}}) \mathrm{d}x.
\end{aligned}
$$

$$
|I_2| \leqslant \frac{1}{16} \int_{B_{\frac{5}{16}}} \left(|\nabla^2 \widehat{\mathbf{u}}|^2 + |\Delta \nabla \widehat{\mathbf{d}}|^2 \right) \eta^2 \mathrm{d}x + C \int_{B_{\frac{5}{16}}} \left(|\nabla \widehat{\mathbf{u}}|^2 + |\nabla^2 \widehat{\mathbf{d}}|^2 \right) |\nabla \eta|^2 \mathrm{d}x,
$$

$$
|I_3| \leqslant \frac{1}{16} \int_{B_{\frac{5}{16}}} \left(|\nabla^2 \widehat{\mathbf{u}}|^2 + |\Delta \nabla \widehat{\mathbf{d}}|^2 \right) \eta^2 \mathrm{d}x + C \int_{B_{\frac{5}{16}}} \left(|\nabla \widehat{\mathbf{u}}|^2 + |\nabla^2 \widehat{\mathbf{d}}|^2 \right) |\nabla \eta|^2 \mathrm{d}x.
$$

Putting these estimates into (4.47), we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_{B_{\frac{5}{16}}} (|\nabla \widehat{\mathbf{u}}|^2 &+ |\nabla^2 \widehat{\mathbf{d}}|^2) \eta^2 \, \mathrm{d}x + \int_{B_{\frac{5}{16}}} \left(|\nabla^2 \widehat{\mathbf{u}}|^2 + |\nabla^3 \widehat{\mathbf{d}}|^2 \right) \eta^2 \mathrm{d}x \\
&\leqslant C \int_{\mathrm{spt}\,\eta} (|\nabla \widehat{\mathbf{u}}|^2 + |\nabla^2 \widehat{\mathbf{d}}|^2 + |\widehat{P}^{(1)}|^2 + |\widehat{\mathbf{u}}|^3 + |\nabla \widehat{P}^{(2)}|^{\frac{3}{2}} + |\nabla^2 \widehat{P}^{(2)}|^{\frac{3}{2}}) \mathrm{d}x.
\end{aligned}
\tag{4.48}
$$

By Fubini's theorem, there exists $t_* \in \left[-(\frac{5}{16})^2, -(\frac{9}{32})^2 \right]$ such that

$$
\int_{B_{\frac{5}{16}}} \left(|\nabla \widehat{\mathbf{u}}|^2 + |\nabla^2 \widehat{\mathbf{d}}|^2 \right) \eta^2(t_*) \mathrm{d}x \leqslant 100 \int_{\mathbb{P}_{\frac{5}{16}}} (|\nabla \widehat{\mathbf{u}}|^2 + |\nabla^2 \widehat{\mathbf{d}}|^2) \eta^2 \, \mathrm{d}x \, \mathrm{d}t.
$$

Integrating (4.48) for $t \in [t_*, 0]$ yields that

$$
\begin{aligned}
\sup_{-\left(\frac{9}{32}\right)^2 \leqslant t \leqslant 0} & \int_{B_{\frac{5}{16}}} (|\nabla \widehat{\mathbf{u}}|^2 + |\nabla^2 \widehat{\mathbf{d}}|^2) \eta^2(t) \mathrm{d}x + \int_{-\left(\frac{9}{32}\right)^2 \leqslant t \leqslant 0} \int_{B_{\frac{5}{16}}} (|\nabla^2 \widehat{\mathbf{u}}|^2 + |\nabla^3 \widehat{\mathbf{d}}|^2) \eta^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\leqslant C \int_{-\frac{1}{4}}^{0} \int_{\mathrm{spt}\,\eta} (|\nabla \widehat{\mathbf{u}}|^2 + |\nabla^2 \widehat{\mathbf{d}}|^2 + |\widehat{P}^{(1)}|^2 + |\widehat{\mathbf{u}}|^3 + |\nabla \widehat{P}^{(2)}|^{\frac{3}{2}} + |\nabla^2 \widehat{P}^{(2)}|^{\frac{3}{2}}) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + C \int_{\mathbb{P}_{\frac{5}{16}}} (|\nabla \widehat{\mathbf{u}}|^2 + |\nabla^2 \widehat{\mathbf{d}}|^2) \eta^2 \mathrm{d}x \, \mathrm{d}t \\
&\leqslant C \int_{\mathbb{P}_{\frac{1}{2}}} \left(|\nabla \widehat{\mathbf{u}}|^2 + |\nabla^2 \widehat{\mathbf{d}}|^2 + |\widehat{\mathbf{u}}|^3 + |\widehat{P}|^{\frac{3}{2}} \right) \mathrm{d}x \, \mathrm{d}t + C.
\end{aligned}
\tag{4.49}
$$

For the pressure P, taking divergence of the equation $(4.18)_1$ yields that for any $-\frac{1}{4} \leqslant t \leqslant 0$,

$$-\Delta \widehat{P}_{x_i} = \operatorname{div}^2 S_\alpha [\Delta \widehat{\mathbf{d}}, \overline{\mathbf{d}}]_{x_i} \quad \text{in } B_{\frac{5}{16}}. \tag{4.50}$$

We have

$$
\begin{aligned}
\int_{\mathbb{P}_{\frac{1}{4}}} |\nabla \widehat{P}|^{\frac{3}{2}} \mathrm{d}x \, \mathrm{d}t &\leqslant C \int_{\mathbb{P}_{\frac{9}{32}}} \left(|S_\alpha[\Delta \widehat{\mathbf{d}}, \overline{\mathbf{d}}]_{x_i}|^{\frac{3}{2}} + |\widehat{P}|^{\frac{3}{2}} \right) \mathrm{d}x \, \mathrm{d}t \\
&\leqslant C \int_{\mathbb{P}_{\frac{9}{32}}} \left(|\nabla^3 \widehat{\mathbf{d}}|^{\frac{3}{2}} + |\widehat{P}|^{\frac{3}{2}} \right) \mathrm{d}x \, \mathrm{d}t \\
&\leqslant C \int_{\mathbb{P}_{\frac{9}{32}}} |\nabla^3 \widehat{\mathbf{d}}|^2 \mathrm{d}x \, \mathrm{d}t + C \int_{\mathbb{P}_{\frac{9}{32}}} |\widehat{P}|^{\frac{3}{2}} \mathrm{d}x \, \mathrm{d}t + C.
\end{aligned}
\tag{4.51}
$$

Now let η be a cut-off function of $B_{\frac{9}{32}}$, i.e., $\eta \equiv 1$ in $B_{\frac{3}{8}}$. Then, by combining (4.49) and (4.51), we obtain

$$
\begin{aligned}
&\sup_{-\left(\frac{1}{4}\right)^2 \leqslant t \leqslant 0} \int_{B_{\frac{1}{4}}} (|\nabla \widehat{\mathbf{u}}|^2 + |\nabla \widehat{\mathbf{d}}|^2) \mathrm{d}x + \int_{\mathbb{P}_{\frac{1}{4}}} \left(|\nabla^2 \widehat{\mathbf{u}}|^2 + |\nabla^3 \widehat{\mathbf{d}}|^2 + |\nabla \widehat{P}|^{\frac{3}{2}} \right) \mathrm{d}x \, \mathrm{d}t \\
&\qquad \leqslant C \int_{\mathbb{P}_{\frac{1}{2}}} \left(|\widehat{\mathbf{u}}|^3 + |\nabla \widehat{\mathbf{u}}|^2 + |\nabla^2 \widehat{\mathbf{d}}|^2 + |\widehat{P}|^{\frac{3}{2}} \right) \mathrm{d}x \, \mathrm{d}t + C.
\end{aligned}
\tag{4.52}
$$

It turns out that we can extend the energy method above to arbitary order. Here we sketch the proof. For nonnegative multiple indices β, γ and δ such that $\gamma = \beta + \delta$ and δ is of order 1, $|\beta| = k$, then $(\nabla^\beta \widehat{\mathbf{u}}, \nabla^\gamma \widehat{\mathbf{d}}, \nabla^\beta \widehat{P})$ satisfies

$$
\begin{cases}
\partial_t(\nabla^\beta \widehat{\mathbf{u}}) + \nabla(\nabla^\beta \widehat{P}) - \Delta(\nabla^\beta \widehat{\mathbf{u}}) = -\nabla \cdot S_\alpha[\Delta(\nabla^\beta \widehat{\mathbf{d}}), \overline{\mathbf{d}}], \\
\operatorname{div}(\nabla^\beta \widehat{\mathbf{u}}) = 0, \\
\partial_t(\nabla^\gamma \widehat{\mathbf{d}}) - \Delta(\nabla^\gamma \widehat{\mathbf{d}}) = T_\alpha[\nabla(\nabla^\gamma \widehat{\mathbf{u}}), \overline{\mathbf{d}}].
\end{cases}
\tag{4.53}
$$

By differentiating $(\widehat{P}^{(1)}, \widehat{P}^{(2)})$ $(k-1)$ times we can estimate

$$\int_{\mathbb{P}_{\frac{1}{2}}} |\nabla^{k-1} \widehat{P}^{(1)}|^2 \mathrm{d}x \, \mathrm{d}t \leqslant C \int_{\mathbb{P}_{\frac{1}{2}}} |\nabla^{k+1} \widehat{\mathbf{d}}|^2 \mathrm{d}x \, \mathrm{d}t, \tag{4.54}$$

and

$$\int_{\mathbb{P}_{\frac{5}{16}}} |\nabla^l \widehat{P}^{(2)}|^{\frac{3}{2}} \mathrm{d}x \, \mathrm{d}t \leqslant C \int_{\mathbb{P}_{\frac{1}{2}}} |\nabla^{k-1} \widehat{P}|^{\frac{3}{2}} \mathrm{d}x \, \mathrm{d}t + C \int_{\mathbb{P}_{\frac{1}{2}}} |\nabla^{k+1} \widehat{\mathbf{d}}|^2 \mathrm{d}x \, \mathrm{d}t + C, \quad l = k, k+1. \tag{4.55}$$

Multiplying $(4.53)_1$ by $(\nabla^\beta \widehat{\mathbf{u}})\eta^2$ and $(4.53)_3$ by $(\nabla^\gamma \widehat{\mathbf{d}})\eta^2$ and integrating the resulting equations over $B_{\frac{1}{2}}$, and by the same calculation and cancellation, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} &\int_{B_{\frac{5}{16}}} (|\nabla^k \widehat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{\mathbf{d}}|^2)\eta^2 \, \mathrm{d}x + \int_{B_{\frac{5}{16}}} (|\nabla^{k+1}\widehat{\mathbf{u}}|^2 + |\nabla^{k+2}\widehat{\mathbf{d}}|^2)\eta^2 \, \mathrm{d}x \\
&\leqslant C \int_{B_{\frac{5}{16}}} \left(|\nabla^k \widehat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{\mathbf{d}}|^2 + |\nabla^{k-1}\widehat{P}^{(1)}|^2 + |\nabla^{k-1}\widehat{\mathbf{u}}|^3 \right. \\
&\qquad\qquad \left. + |\nabla^k \widehat{P}^{(2)}|^{\frac{3}{2}} + |\nabla^{k+1}\widehat{P}^{(2)}|^{\frac{3}{2}} \right) \mathrm{d}x \\
&\leqslant C \int_{\mathbb{P}_{\frac{1}{2}}} \left(|\nabla^k \widehat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{\mathbf{d}}|^2 + |\nabla^{k-1}\widehat{\mathbf{u}}|^3 + |\nabla^{k-1}\widehat{P}|^{\frac{3}{2}} \right) \mathrm{d}x \, \mathrm{d}t + C.
\end{aligned}
\tag{4.56}
$$

For P, since

$$
-\Delta(\nabla^\beta \widehat{P}) = \mathrm{div}^2 \, S_\alpha[\Delta(\nabla^\beta \widehat{\mathbf{u}}), \overline{\mathbf{d}}] \quad \text{in } B_{\frac{5}{16}}, \tag{4.57}
$$

we have

$$
\begin{aligned}
\int_{\mathbb{P}_{\frac{1}{4}}} |\nabla^k \widehat{P}|^{\frac{3}{2}} \, \mathrm{d}x \, \mathrm{d}t &\leqslant C \int_{\mathbb{P}_{\frac{9}{32}}} |\nabla^{k+2}\widehat{\mathbf{d}}|^{\frac{3}{2}} \, \mathrm{d}x \, \mathrm{d}t + C \int_{\mathbb{P}_{\frac{9}{32}}} |\nabla^{k-1}\widehat{P}|^{\frac{3}{2}} \, \mathrm{d}x \, \mathrm{d}t \\
&\leqslant C \int_{\mathbb{P}_{\frac{9}{32}}} |\nabla^{k+2}\widehat{\mathbf{d}}|^2 \, \mathrm{d}x \, \mathrm{d}t + C \int_{\mathbb{P}_{\frac{9}{32}}} |\nabla^{k-1}\widehat{P}|^{\frac{3}{2}} \, \mathrm{d}x \, \mathrm{d}t + C.
\end{aligned}
\tag{4.58}
$$

By choosing suitable t_* as above, we can integrate (4.56) in t to get

$$
\begin{aligned}
\sup_{-\left(\frac{9}{32}\right)^2 \leqslant t \leqslant 0} &\int_{B_{\frac{9}{32}}} (|\nabla^k \widehat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{\mathbf{d}}|^2) \mathrm{d}x + \int_{\mathbb{P}_{\frac{9}{32}}} (|\nabla^{k+1}\widehat{\mathbf{u}}|^2 + |\nabla^{k+2}\widehat{\mathbf{d}}|^2) \, \mathrm{d}x \, \mathrm{d}t \\
&\leqslant C \int_{\mathbb{P}_{\frac{1}{2}}} \left(|\nabla^k \widehat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{\mathbf{d}}|^2 + |\nabla^{k-1}\widehat{\mathbf{u}}|^3 + |\nabla^{k-1}\widehat{P}|^{\frac{3}{2}} \right) \mathrm{d}x \, \mathrm{d}t + C.
\end{aligned}
\tag{4.59}
$$

Thus, we get

$$
\begin{aligned}
\sup_{-\left(\frac{1}{4}\right)^2 \leqslant t \leqslant 0} &\int_{B_{\frac{1}{4}}} (|\nabla^k \widehat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{\mathbf{d}}|^2) \mathrm{d}x \\
&\quad + \int_{\mathbb{P}_{\frac{1}{4}}} \left(|\nabla^{k+1}\widehat{\mathbf{u}}|^2 + |\nabla^{k+2}\widehat{\mathbf{d}}|^2 + |\nabla^k \widehat{P}|^{\frac{3}{2}} \right) \mathrm{d}x \, \mathrm{d}t \\
&\leqslant C \int_{\mathbb{P}_{\frac{1}{2}}} \left(|\nabla^{k-1}\widehat{\mathbf{u}}|^3 + |\nabla^k \widehat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{\mathbf{d}}|^2 + |\nabla^{k-1}\widehat{P}|^{\frac{3}{2}} \right) \mathrm{d}x \, \mathrm{d}t + C.
\end{aligned}
\tag{4.60}
$$

From Sobolev's interpolation inequality, we have

$$\int_{\mathbb{P}_{\frac{1}{2}}} |\nabla^{k-1}\widehat{\mathbf{u}}|^3 \mathrm{d}x\,\mathrm{d}t \leqslant C\big\|\nabla^{k-1}\widehat{\mathbf{u}}\big\|^6_{L^\infty_t L^2_x\left(\mathbb{P}_{\frac{1}{2}}\right)} + C\int_{\mathbb{P}_{\frac{1}{2}}} (|\nabla^{k-1}\widehat{\mathbf{u}}|^2 + |\nabla^k\widehat{\mathbf{u}}|^2)\,\mathrm{d}x\,\mathrm{d}t.$$

Substituting this inequality in (4.60) and by suitable adjusting of the radius, we can show that

$$
\begin{aligned}
&\sup_{-\left(\frac{1}{4}\right)^2 \leqslant t \leqslant 0} \int_{B_{\frac{1}{4}} \times \{t\}} (|\nabla^k\widehat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{\mathbf{d}}|^2)\mathrm{d}x \\
&\quad + \int_{\mathbb{P}_{\frac{1}{4}}} \left(|\nabla^{k+1}\widehat{\mathbf{u}}|^2 + |\nabla^{k+2}\widehat{\mathbf{d}}|^2 + |\nabla^k\widehat{P}|^{\frac{3}{2}} \right) \mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant C\left(\|(\widehat{\mathbf{u}}, \nabla\widehat{\mathbf{d}})\|_{L^\infty_t L^2_x \cap L^2_t H^2_x\left(\mathbb{P}_{\frac{1}{2}}\right)}, \|\widehat{P}\|_{L^{\frac{3}{2}}\left(\mathbb{P}_{\frac{1}{2}}\right)} \right).
\end{aligned}
\tag{4.61}
$$

With (4.61), we can apply the regularity for both the linear Stokes equations and the linear heat equation (cf [16, 21]) to conclude that $(\widehat{\mathbf{u}}, \widehat{\mathbf{d}}) \in C^\infty(\mathbb{P}_{\frac{1}{4}})$. Furthermore, applying the elliptic estimate for the pressure equation (4.40), we see that $\widehat{P} \in C^\infty(\mathbb{P}_{\frac{1}{4}})$. Therefore $(\widehat{\mathbf{u}}, \widehat{\mathbf{d}}, \widehat{P}) \in C^\infty(\mathbb{P}_{\frac{1}{4}})$ and the estimate (4.39) holds. The proof is completed. $\qquad\square$

The oscillation lemma admits the following iterations.

Lemma 4.4. *Let* $(\mathbf{u}, \mathbf{d}, P), M, \varepsilon_0(M), \tau_0(M), C_0(M), z_0$ *be as in lemma* 4.1. *Then there exist* $r_0 = r_0(M), \varepsilon_1 = \varepsilon_1(M) > 0$ *such that for* $0 < r \leqslant r_0$, *if*

$$|\mathbf{d}_{z_0,r}| \leqslant \frac{M}{2}, \qquad \Phi(z_0, r) \leqslant \varepsilon_1^3,$$

then for any $k = 1, 2, \dots$, *we have*

$$
\begin{aligned}
&|\mathbf{d}_{z_0, \tau_0^{k-1}r}| \leqslant M, \\
&\Phi(z_0, \tau_0^{k-1}r) \leqslant \varepsilon_1^3, \\
&\Phi(z_0, \tau_0^k r) \leqslant \frac{1}{2}\, \max\left\{ \Phi(z_0, \tau_0^{k-1}r), C_0(\tau_0^{k-1}r)^3 \right\}.
\end{aligned}
\tag{4.62}
$$

Proof. We prove it by an induction on k. By translational invariance we may assume that $z_0 = 0$, and we abbreviate $\mathbf{d}_{0,r}$ to $\mathbf{d}_r$ for simplicity.

For $k = 1$, the conclusion follows from lemma 4.1, if we choose ε_1 such that $\varepsilon_1 < \varepsilon_0$. Suppose the conclusion is true for all $k \leqslant k_0, k_0 \geqslant 1$, we show it remains true for $k = k_0 + 1$. By the inductive hypothesis

$$
\begin{aligned}
&|\mathbf{d}_{\tau_0^{k-1}r}| \leqslant M, \\
&\Phi(0, \tau_0^{k-1}r) \leqslant \varepsilon_1^3, \\
&\Phi(0, \tau_0^k r) \leqslant \frac{1}{2}\, \max\left\{ \Phi(0, \tau_0^{k-1}r), C_0(\tau_0^{k-1}r)^3 \right\} \leqslant \frac{1}{2}\, \max\left\{ \varepsilon_1^3, C_0(\tau_0^{k-1}r)^3 \right\}
\end{aligned}
$$

for all $k \leqslant k_0$. Thus,

$$
\begin{aligned}
\Phi(0, \tau_0^k r) &\leqslant \frac{1}{2} \max\left\{ \Phi(0, \tau_0^{k-1} r), C_0(\tau_0^{k-1} r)^3 \right\} \\
&\leqslant \frac{1}{2} \max\left\{ \frac{1}{2} \max\left\{ \Phi(0, \tau_0^{k-2} r), C_0(\tau_0^{k-2} r)^3 \right\}, C_0(\tau_0^{k-1} r)^3 \right\} \\
&\leqslant \cdots \leqslant 2^{-k} \max\left\{ \Phi(0, r), \frac{C_0 r^3}{1 - 2\tau_0^3} \right\} \\
&\leqslant 2^{-k} \max\left\{ \varepsilon_1^3, \frac{C_0 r_0^3}{1 - 2\tau_0^3} \right\}, \ \forall \, k \leqslant k_0.
\end{aligned}
$$

Then

$$
\begin{aligned}
|\mathbf{d}_{\tau_0^{k_0} r}| &\leqslant |\mathbf{d}_r| + \sum_{k=1}^{k_0} |\mathbf{d}_{\tau_0^k} - \mathbf{d}_{\tau_0^{k-1} r}| \\
&\leqslant \frac{M}{2} + \sum_{k=1}^{k_0} \left(\fint_{\mathbb{P}_{\tau_0^k r}(0)} |\mathbf{d} - \mathbf{d}_{\tau_0^{k-1} r}|^6 \right)^{\frac{1}{6}} \\
&\leqslant \frac{M}{2} + \sum_{k=1}^{k_0} \Phi(0, \tau_0^{k-1} r)^{\frac{1}{3}} \\
&\leqslant \frac{M}{2} + \sum_{k=1}^{k_0} 2^{-\frac{1}{3}(k-1)} \max\left\{ \varepsilon_1, \left(\frac{C_0 r_0^3}{1 - 2\tau_0^3} \right)^{\frac{1}{3}} \right\} \\
&\leqslant \frac{M}{2} + \frac{1}{1 - 2^{-\frac{1}{3}}} \max\left\{ \varepsilon_1, \left(\frac{C_0 r_0^3}{1 - 2\tau_0^3} \right)^{\frac{1}{3}} \right\}.
\end{aligned}
$$

If we choose sufficiently small $r_0 = r_0(M)$, $\varepsilon_1 = \varepsilon_1(M)$, we see

$$
|\mathbf{d}_{\tau_0^{k_0} r}| \leqslant M,
$$
$$
\Phi(0, \tau_0^{k_0} r) \leqslant \varepsilon_1^3 \leqslant \varepsilon_0^3.
$$

It follows directly from lemma 4.1 with r replaced by $\tau_0^k r$ that

$$
\Phi(0, \tau_0^{k+1} r) \leqslant \frac{1}{2} \max\left\{ \Phi(0, \tau_0^k r), C_0(\tau_0^k r)^3 \right\}.
$$

This completes the proof. □

The local boundedness of the solutions can be obtained by utilizing the Riesz potential estimates between Morrey spaces as in the following lemma.

Lemma 4.5. *For any $M > 0$, there exists $\varepsilon_2 > 0$, depending on M, such that if $(\mathbf{u}, \mathbf{d}, P)$ is a suitable weak solution of* (1.2) *in $\mathbb{R}^3 \times (0, \infty)$, which satisfies, for $z_0 = (x_0, t_0) \in \mathbb{R}^3 \times (r_0^2, \infty)$*

$$
|\mathbf{d}_{z_0, r_0}| \leqslant \frac{M}{4}, \quad and \quad \Phi(z_0, r_0) \leqslant \varepsilon_2^3, \tag{4.63}
$$

then for any $1 < p < \infty$, $(\mathbf{u}, \nabla\mathbf{d}) \in L^p(\mathbb{P}_{\frac{r_0}{4}}(z_0))$, $\mathbf{d} \in C^\theta(\mathbb{P}_{\frac{r_0}{2}}(z_0))$ *and*

$$|\mathbf{d}| \leqslant M \quad in \ \mathbb{P}_{\frac{r_0}{2}}(z_0), \qquad [\mathbf{d}]_{C^\theta(\mathbb{P}_{\frac{r_0}{2}}(z_0))} \leqslant C(\theta, M)(\varepsilon_1 + r_0). \tag{4.64}$$

$$\|(\mathbf{u}, \nabla\mathbf{d})\|_{L^p(\mathbb{P}_{\frac{r_0}{4}}(z_0))} \leqslant C(p, M)(\varepsilon_1 + r_0), \tag{4.65}$$

where ε_1 is the constant in lemma 4.4.

Proof. Let $\varepsilon_2 = \min\left\{\left(\frac{M}{4}\right), 2^{-\frac{11}{6}}\varepsilon_1(M)\right\}$. For any $z \in \mathbb{P}_{\frac{r_0}{2}}(z_0)$,

$$\begin{aligned}
|\mathbf{d}_{z,\frac{r_0}{2}}| &\leqslant |\mathbf{d}_{z,\frac{r_0}{2}} - \mathbf{d}_{z_0,r_0}| + |\mathbf{d}_{z_0,r_0}| \\
&\leqslant \fint_{\mathbb{P}_{\frac{r_0}{2}}(z)} |\mathbf{d} - \mathbf{d}_{z_0,r_0}| + \frac{M}{4} \leqslant \varepsilon_2 + \frac{M}{4} \leqslant \frac{M}{2}.
\end{aligned}$$

Meanwhile,

$$\left(\fint_{\mathbb{P}_{\frac{r_0}{2}}(z)} |\mathbf{d} - \mathbf{d}_{z,\frac{r_0}{2}}|^6 \mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}$$

$$\leqslant \left(2^5 \fint_{\mathbb{P}_{\frac{r_0}{2}}(z)} |\mathbf{d} - \mathbf{d}_{z_0,r_0}|^6 \mathrm{d}x\,\mathrm{d}t + 2^5 |\mathbf{d}_{z_0,r_0} - \mathbf{d}_{z,\frac{r_0}{2}}|^6\right)^{\frac{1}{2}}$$

$$\leqslant \left(2^{10} \fint_{\mathbb{P}_{r_0}(z_0)} |\mathbf{d} - \mathbf{d}_{z_0,r_0}|^6 \mathrm{d}x\,\mathrm{d}t + 2^5 \fint_{\mathbb{P}_{\frac{r_0}{2}}(z)} |\mathbf{d} - \mathbf{d}_{z_0,r_0}|^6 \mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}$$

$$\leqslant 2^{\frac{11}{2}} \left(\fint_{\mathbb{P}_{r_0}(z_0)} |\mathbf{d} - \mathbf{d}_{z_0,r_0}|^6 \mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}},$$

Hence we get that

$$\Phi\left(z, \frac{r_0}{2}\right) \leqslant 2^{\frac{11}{2}} \Phi(z_0, r_0) \leqslant 2^{\frac{11}{2}} \varepsilon_2^3 \leqslant \varepsilon_1^3.$$

Then we deduce from lemma 4.4 that for any $k = 1, 2, \ldots$,

$$\begin{aligned}
&|\mathbf{d}_{z,\tau_0^{k-1}\frac{r_0}{2}}| \leqslant M, \\
&\Phi\left(z, \tau_0^k \frac{r_0}{2}\right) \leqslant \frac{1}{2} \max\left\{\Phi\left(z, \tau_0^{k-1}\frac{r_0}{2}\right), C_0(\tau_0^{k-1}r)^3\right\}.
\end{aligned} \tag{4.66}$$

By Lebesgue's differentiation theorem, we have $|d| \leqslant M$ a.e. in $\mathbb{P}_{\frac{r_0}{2}}(z_0)$. Furthermore, we have

$$\Phi\left(z, \frac{\tau_0^k r_0}{2}\right) \leqslant 2^{-k} \max\left\{\Phi\left(z, \frac{r_0}{2}\right), \frac{C_0 r_0^3}{1 - 2\tau_0^3}\right\}.$$

Therefore for $\theta_0 = \frac{\ln 2}{3|\ln \tau_0|} \in (0, \frac{1}{3})$, it holds for any $0 < s < \frac{r_0}{2}$ and $z \in \mathbb{P}_{\frac{r_0}{2}}(z_0)$,

$$\Phi(z, s) \leqslant C(r_0^3 + \varepsilon_1^3)\left(\frac{s}{r_0}\right)^{3\theta_0}. \tag{4.67}$$

By the Campanato theory, $\mathbf{d} \in C^\theta(\mathbb{P}_{\frac{r_0}{4}}(z_0))$ and (4.65) holds. Now for $\phi \in C_0^\infty(\mathbb{P}_{\frac{r_0}{2}})(z_0)$, from (2.3) and (2.4) we can derive the following local energy inequality:

$$
\frac{1}{2}\int_{\mathbb{R}^3}(|\mathbf{u}|^2 + |\nabla\mathbf{d}|^2)\phi(x,t)\mathrm{d}x + \int_0^t\int_{\mathbb{R}^3}(|\nabla\mathbf{u}|^2 + |\Delta\mathbf{d}|^2)\phi(x,s)\,\mathrm{d}x\,\mathrm{d}s
$$

$$
\leqslant \int_0^t\int_{\mathbb{R}^3}\frac{1}{2}(|\mathbf{u}|^2 + |\nabla\mathbf{d}|^2)(\partial_t\phi + \Delta\phi)(x,s)\,\mathrm{d}x\,\mathrm{d}s
$$

$$
+ \int_0^t\int_{\mathbb{R}^3}\left[\frac{1}{2}(|\mathbf{u}|^2 + 2P)\mathbf{u}\cdot\nabla\phi + \nabla\mathbf{d}\odot\nabla\mathbf{d} : \mathbf{u}\otimes\nabla\phi\right](x,s)\,\mathrm{d}x\,\mathrm{d}s
$$

$$
+ \int_0^t\int_{\mathbb{R}}^3(\nabla\mathbf{d}\odot\nabla\mathbf{d} - |\nabla\mathbf{d}|^2 I_3) : \nabla^2\phi(x,s)\,\mathrm{d}x\,\mathrm{d}s
$$

$$
+ \int_0^t\int_{\mathbb{R}^3}[S_\alpha[\Delta\mathbf{d},\mathbf{d}] : \mathbf{u}\otimes\nabla\phi + T_\alpha[\nabla\mathbf{u},\mathbf{d}]\cdot(\nabla\phi\cdot\nabla\mathbf{d})](x,s)\,\mathrm{d}x\,\mathrm{d}s
$$

$$
+ \int_0^t\int_{\mathbb{R}^3}\nabla\cdot S_\alpha[\mathbf{f}(\mathbf{d}),\mathbf{d}]\cdot\mathbf{u}\phi(x,s)\,\mathrm{d}x\,\mathrm{d}s
$$

$$
+ \int_0^t\int_{\mathbb{R}^3}(\mathbf{u}\cdot\nabla\mathbf{d})\cdot\mathbf{f}(\mathbf{d})\phi(x,s)\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_{\mathbb{R}^3}\nabla\mathbf{f}(\mathbf{d}) : \nabla\mathbf{d}\phi(x,s)\,\mathrm{d}x\,\mathrm{d}s.
$$

$$
(4.68)
$$

Let $\phi \in C_0^\infty(\mathbb{P}_{2s}(z))$ be a cut-off function of $\mathbb{P}_s(z)$. Replacing ϕ by ϕ^2 in (4.68), we can show that for $0 < s < \frac{r_0}{2}$,

$$
s^{-1}\int_{\mathbb{P}_s(z)}(|\nabla\mathbf{u}|^2 + |\Delta\mathbf{d}|^2)\,\mathrm{d}x\,\mathrm{d}t
$$

$$
\leqslant C\left[(2s)^{-3}\int_{\mathbb{P}_{2s}(z)}(|\mathbf{u}|^2 + |\nabla\mathbf{d}|^2)\,\mathrm{d}x\,\mathrm{d}t\right.
$$

$$
+ (2s)^{-2}\int_{\mathbb{P}_{2s}(z)}\left(|\mathbf{u}|^3 + |\nabla\mathbf{d}|^3 + |P|^{\frac{3}{2}}\right)\mathrm{d}x\,\mathrm{d}t\right]
$$

$$
(4.69)
$$

$$
\leqslant C(r_0^3 + \varepsilon_1^3)\left(\frac{s}{r_0}\right)^{2\theta_0}.
$$

Now we are ready to perform the Riesz potential estimate. For any open set $U \subset \mathbb{R}^3\times\mathbb{R}$, $1 \leqslant p < \infty$, define the Morrey space $M^{p,\lambda}(U)$ by

$$
M^{p,\lambda}(U) := \left\{f \in L_{\mathrm{loc}}^p(U) : \|f\|_{M^{p,\lambda}(U)}^p = \sup_{z\in U, r>0} r^{\lambda-5}\int_{\mathbb{P}_r(z)}|f|^p\mathrm{d}x\,\mathrm{d}t < \infty\right\}.
$$

It follows from (4.67) and (4.69) that there exists $\alpha \in (0,1)$ such that

$$
(\mathbf{u}, \nabla\mathbf{d}) \in M^{3,3(1-\alpha)}\left(\mathbb{P}_{\frac{r_0}{2}}(z_0)\right), P \in M^{\frac{3}{2},3(1-\alpha)}\left(\mathbb{P}_{\frac{r_0}{2}}(z_0)\right),
$$

$$
(\nabla\mathbf{u}, \nabla^2\widehat{\mathbf{d}}) \in M^{2,4-2\alpha}\left(\mathbb{P}_{\frac{r_0}{2}}(z_0)\right).
$$

Write **d** equation in (1.2) as

$$\partial_t \mathbf{d} - \Delta \mathbf{d} = -\mathbf{u} \cdot \nabla \mathbf{d} + T_\alpha[\nabla \mathbf{u}, \mathbf{d}] - \mathbf{f}(\mathbf{d}) \in M^{\frac{3}{2}, 3(1-\alpha)}\left(\mathbb{P}_{\frac{r_0}{2}}(z_0)\right). \qquad (4.70)$$

Let $\eta \in C_0^\infty(\mathbb{R}^4)$ be such that $0 \leqslant \eta \leqslant 1$, $\eta = 1$ in $\mathbb{P}_{\frac{r_0}{2}}(z_0)$, $|\partial_t \eta| + |\nabla^2 \eta| \leqslant Cr_0^2$. Set $\mathbf{w} = \eta^2 (\mathbf{d} - \mathbf{d}_{z_0, \frac{r_0}{2}})$. Then

$$\partial_t \mathbf{w} - \Delta \mathbf{w} = F, \quad F := \eta^2 (\partial_t \mathbf{d} - \Delta \mathbf{d}) + (\partial_t \eta^2 - \Delta \eta^2)\left(\mathbf{d} - \mathbf{d}_{z_0, \frac{r_0}{2}}\right) - 2\nabla \eta^2 \cdot \nabla \mathbf{d}.$$
$$(4.71)$$

We can check that $F \in M^{\frac{3}{2}, 3(1-\alpha)}(\mathbb{R}^4)$ and satisfies

$$\|F\|_{M^{\frac{3}{2}, 3(1-\alpha)}(\mathbb{R}^4)} \leqslant C(r_0 + \varepsilon_1). \qquad (4.72)$$

Let Γ denote the heat kernel in $\mathbb{R}^3$. Then

$$|\nabla \Gamma(x, t)| \leqslant C\delta^{-4}((x, t), (0, 0)), \forall (x, t) \neq (0, 0),$$

where $\delta(\cdot, \cdot)$ denotes the parabolic distance on $\mathbb{R}^4$. By the Duhamel formula, we have that

$$|\mathbf{w}(x, t)| \leqslant \int_0^t \int_{\mathbb{R}^3} |\nabla \Gamma(x - y, t - s)| |F(y, s)| \mathrm{d}y \, \mathrm{d}s \leqslant C\mathcal{I}_1(|F|)(x, t), \qquad (4.73)$$

where $\mathcal{I}_\beta$ is the parabolic Riesz potential of order β on $\mathbb{R}^4$, $0 \leqslant \beta \leqslant 5$, defined by

$$\mathcal{I}_\beta(g)(x, t) = \int_{\mathbb{R}^4} \frac{|g(y, s)|}{\delta^{5-\beta}((x, t), (y, s))} \mathrm{d}y \, \mathrm{d}s, \ \forall \ g \in L^2(\mathbb{R}^4).$$

Applying the Riesz potential estimates [12], we conclude that $\nabla \mathbf{w} \in M^{\frac{3(1-\alpha)}{1-2\alpha}, 3(1-\alpha)}(\mathbb{R}^4)$ and

$$\|\nabla \mathbf{w}\|_{M^{\frac{3(1-\alpha)}{1-2\alpha}, 3(1-\alpha)}(\mathbb{R}^4)} \leqslant C\|F\|_{M^{\frac{3}{2}, 3(1-\alpha)}(\mathbb{R}^4)} \leqslant C(r_0 + \varepsilon_1). \qquad (4.74)$$

Since $\lim_{\alpha \uparrow \frac{1}{2}} \frac{3(1-\alpha)}{1-2\alpha} = \infty$, we conclude that for any $1 < p < \infty$, $\nabla \mathbf{w} \in L^p(\mathbb{P}_{r_0}(z_0))$ and

$$\|\nabla \mathbf{w}\|_{L^p(\mathbb{P}_{\frac{r_0}{2}}(z_0))} \leqslant C(p)(r_0 + \varepsilon_1).$$

Since $\mathbf{d} - \mathbf{w}$ solves

$$\partial_t (\mathbf{d} - \mathbf{w}) - \Delta(\mathbf{d} - \mathbf{w}) = 0 \quad \text{in } \mathbb{P}_{\frac{r_0}{4}}(z_0),$$

it follows from the theory of heat equations that $\nabla(\mathbf{d} - \mathbf{w}) \in L^\infty(\mathbb{P}_{\frac{r_0}{4}}(z_0))$. Therefore for any $1 < p < \infty$, $\mathbf{d} \in L^p(\mathbb{P}_{\frac{r_0}{4}}(z_0))$, and

$$\|\nabla \mathbf{d}\|_{L^p(\mathbb{P}_{\frac{r_0}{4}}(z_0))} \leqslant C(p)(r_0 + \varepsilon_1).$$

We now proceed with the estimation of $\mathbf{u}$. Let $\mathbf{v} : \mathbb{R}^3 \times (0, \infty) \mapsto \mathbb{R}^3$ solve the Stokes equation:

$$
\begin{cases}
\partial_t \mathbf{v} - \Delta \mathbf{v} + \nabla P = -\operatorname{div}\left[\eta^2\left(\mathbf{u} \otimes \mathbf{u} + \nabla\mathbf{d} \odot \nabla\mathbf{d} - \frac{1}{2}|\nabla\mathbf{d}|^2 I_3\right)\right] \\
\qquad\qquad\qquad\qquad + \operatorname{div}\left\{\eta^2\left(F(\mathbf{d}) - F(\mathbf{d})_{z_0,\frac{r_0}{2}}\right)I_3\right\} \\
\qquad\qquad\qquad\qquad - \operatorname{div}\left\{\eta^2\left(S_\alpha[\Delta\mathbf{d} - \mathbf{f}(\mathbf{d}), \mathbf{d}] + S_\alpha[\mathbf{f}(\mathbf{d}), \mathbf{d}]_{z_0,\frac{r_0}{2}}\right)\right\}, \\
\nabla \cdot \mathbf{v} = 0, \\
\mathbf{v}(\cdot, 0) = 0.
\end{cases}
\tag{4.75}
$$

By using the Oseen kernel, an estimate of $\mathbf{v}$ can be given by

$$
|\mathbf{v}(x, t)| \leqslant C\mathcal{I}_1(|X|)(x, t), \forall (x, t) \in \mathbb{R}^3 \times (0, \infty),
\tag{4.76}
$$

where

$$
\begin{aligned}
X = \eta^2 &\left[\mathbf{u} \otimes \mathbf{u} + \left(\nabla\mathbf{d} \odot \nabla\mathbf{d} - \frac{1}{2}|\nabla\mathbf{d}|^2 I_3\right) - (F(\mathbf{d}) - F(\mathbf{d})_{z_0,\frac{r_0}{2}})I_3 \right. \\
&\left. + S_\alpha[\Delta\mathbf{d} - \mathbf{f}(\mathbf{d}), \mathbf{d}] + S_\alpha[\mathbf{f}(\mathbf{d}), \mathbf{d}]_{z_0,\frac{r_0}{2}}\right].
\end{aligned}
$$

As above, we can check that $X \in M^{\frac{3}{2}, 3(1-\alpha)}(\mathbb{R}^4)$ and

$$
\|X\|_{M^{\frac{3}{2}, 3(1-\alpha)}(\mathbb{R}^4)} \leqslant C(r_0 + \varepsilon_1).
$$

Hence we conclude that $\mathbf{v} \in M^{\frac{3(1-\alpha)}{1-2\alpha}, 3(1-\alpha)}(\mathbb{R}^4)$, and

$$
\|\mathbf{v}\|_{M^{\frac{3(1-\alpha)}{1-2\alpha}, 3(1-\alpha)}(\mathbb{R}^4)} \leqslant C\|X\|_{M^{\frac{3}{2}, 3(1-\alpha)}(\mathbb{R}^4)} \leqslant C(r_0 + \varepsilon_1).
\tag{4.77}
$$

As $\alpha \uparrow \frac{1}{2}$, $\frac{3(1-\alpha)}{1-2\alpha} \to \infty$, we conclude that for any $1 < p < \infty$, $\mathbf{v} \in L^p(\mathbb{P}_{\frac{r_0}{2}}(z_0))$. Since

$$
\partial_t(\mathbf{u} - \mathbf{v}) - \Delta(\mathbf{u} - \mathbf{v}) + \nabla P = 0, \quad \operatorname{div}(\mathbf{u} - \mathbf{v}) = 0 \quad \text{in } \mathbb{P}_{\frac{r_0}{2}(z_0)},
$$

we have that $\mathbf{u} - \mathbf{v} \in L^\infty(\mathbb{P}_{\frac{r_0}{4}}(z_0))$. Therefore for any $1 < p < \infty$, $\mathbf{u} \in L^p(\mathbb{P}_{\frac{r_0}{4}}(z_0))$ and

$$
\|\mathbf{u}\|_{L^p(\mathbb{P}_{\frac{r_0}{4}}(z_0))} \leqslant C(p)(r_0 + \varepsilon_1).
$$

$\square$

For the rest of this section, we will establish the higher order regularity of (1.2). Again we prove it via a high order energy method which has been employed by Huang–Lin–Wang [11]

for general Ericksen–Leslie systems in dimension two, and Du–Hu–Wang [6] for co-rotational Beris–Edwards model in dimension three.

Lemma 4.6. *Under the same assumption as lemma* 4.5, *we have that for any* $k \geqslant 0$, $(\nabla^k \mathbf{u}, \nabla^{k+1} \mathbf{d}) \in (L_t^\infty L_x^2 \cap L_t^2 H_x^1) \left(\mathbb{P}_{\frac{1+2^{-(k+1)}}{2} r_0}(z_0) \right)$ *and the following estimates hold*

$$
\sup_{t_0 - \left(\frac{1+2^{-(k+1)}}{2} r_0 \right)^2 \leqslant t \leqslant t_0} \int_{B_{\frac{1+2^{-(k+1)}}{2} r_0}(x_0)} (|\nabla^k \mathbf{u}|^2 + |\nabla^{k+1} \mathbf{d}|^2) \mathrm{d}x
$$

$$
+ \int_{\mathbb{P}_{\frac{1+2^{-(k+1)}}{2} r_0}(z_0)} \left(|\nabla^{k+1} \mathbf{u}|^2 + |\nabla^{k+2} \mathbf{d}|^2 + |\nabla^k P|^{\frac{3}{2}} \right) \mathrm{d}x \, \mathrm{d}t \tag{4.78}
$$

$$
\leqslant C(k, r_0)\varepsilon_1.
$$

In particular, $(\widehat{\mathbf{u}}, \widehat{\mathbf{d}})$ *is smooth in* $\mathbb{P}_{\frac{r_0}{4}}(z_0)$.

Proof. For simplicity, assume $z_0 = (0, 0)$ and $r_0 = 2$. (4.78) can be proved by an induction on k. It is clear that when $k = 0$, (4.78) follows directly from the local energy inequality (4.68). Here we indicate to how to proof (4.78) for $k \geqslant 1$. Suppose that (4.78) holds for $k \leqslant l - 1$, we want to show that (4.78) also holds for $k = l$. From the induction hypothesis, we have that for $0 \leqslant k \leqslant l - 1$,

$$
\sup_{-(1+2^{-(k+1)})^2 \leqslant t \leqslant 0} \int_{B_{1+2^{-(k+1)}}} (|\nabla^k \mathbf{u}|^2 + |\nabla^{k+1} \mathbf{d}|^2) \mathrm{d}x
$$

$$
+ \int_{\mathbb{P}_{1+2^{-(k+1)}}} \left(|\nabla^{k+1} \mathbf{u}| + |\nabla^{k+2} \mathbf{d}|^2 + |\nabla^k P|^{\frac{3}{2}} \right) \mathrm{d}x \, \mathrm{d}t \leqslant C(l)\varepsilon_1. \tag{4.79}
$$

Hence by the Sobolev embedding we have

$$
\int_{\mathbb{P}_{1+2^{-l}}} \left(|\nabla^{l-1} \mathbf{u}|^{\frac{10}{3}} + |\nabla^l \mathbf{d}|^{\frac{10}{3}} \right) \mathrm{d}x \, \mathrm{d}t \leqslant C(l)\varepsilon_1, \tag{4.80}
$$

and for $0 \leqslant k \leqslant l - 2$, by the Sobolev-interpolation inequality as in (3.11) we have

$$
\int_{\mathbb{P}_{1+2^{-(k+1)}}} (|\nabla^k \mathbf{u}|^{10} + |\nabla^{k+1} \mathbf{d}|^{10}) \mathrm{d}x \, \mathrm{d}t \leqslant C(l)\varepsilon_1. \tag{4.81}
$$

Also, for $1 \leqslant j \leqslant l - 1$, we have

$$
\int_{-(1+2^{-j})^2}^0 \left\| (\nabla^j \mathbf{u}, \nabla^{j+1} \mathbf{d}) \right\|_{L^3(B_{1+2^{-j}})}^4 \mathrm{d}t
$$

$$
\leqslant \int_{-(1+2^{-j})^2}^0 \left\| (\nabla^j \mathbf{u}, \nabla^{j+1} \mathbf{d}) \right\|_{L^2(B_{1+2^{-j}})}^2 \left\| (\nabla^j \mathbf{u}, \nabla^{j+1} \mathbf{d}) \right\|_{L^6(B_{1+2^{-j}})}^2 \mathrm{d}t \tag{4.82}
$$

$$
\leqslant \left\| (\nabla^j \mathbf{u}, \nabla^{j+1} \mathbf{d}) \right\|_{L_t^\infty L_x^2(\mathbb{P}_{1+2^{-j}})}^2 \left\| (\nabla^j \mathbf{u}, \nabla^{j+1} \mathbf{d}) \right\|_{L_t^2 H_x^1(\mathbb{P}_{1+2^{-j}})}^2 \leqslant C(l)\varepsilon_1.
$$

By lemma 4.5 we also have that any $i \in \mathbb{N}^+$ and $1 < p < \infty$,

$$
\|\mathbf{d}\|_{L^\infty(\mathbb{P}_2)} \leqslant M, \quad [\mathbf{d}]_{C^\theta(\mathbb{P}_2)} + [D_\mathbf{d}^i \mathbf{f}(\mathbf{d})]_{C^\theta(\mathbb{P}_2)} \leqslant C(i, M)\varepsilon_0,
$$
$$
\|(\mathbf{u}, \nabla \mathbf{d})\|_{L^p(\mathbb{P}_2)} \leqslant C(p)\varepsilon_1.
$$
(4.83)

Notice that $\nabla^{l-1} P$ satisfies

$$
-\Delta \nabla^{l-1} P = \operatorname{div}^2 \left[\nabla^{l-1} \left(\mathbf{u} \otimes \mathbf{u} + \nabla \mathbf{d} \odot \nabla \mathbf{d} - \frac{1}{2}|\nabla \mathbf{d}|^2 I_3 - (F(\mathbf{d})I_3 \right. \right.
$$
$$
\left. \left. - \fint_{\mathbb{P}_2} F(\mathbf{d})I_3 \right) + S_\alpha[\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}), \mathbf{d}] + \fint_{\mathbb{P}_2} S_\alpha[\mathbf{f}(\mathbf{d}), \mathbf{d}] \right) \right].
$$
(4.84)

Now let $\zeta \in C_0^\infty(B_{1+2^{-l}})$ be a cut-off function of $B_{1+2^{-(l+1)}+3^{-(l+1)}}$, and $P^{(1)}(\cdot, t): \mathbb{R}^3 \to \mathbb{R}$, $-(1 + 2^{-1})^2 \leqslant t \leqslant 0$,

$$
P^{(1)}(x, t) := \int_{\mathbb{R}^3} \nabla_x^2 G(x - y)\zeta(y) \left[\mathbf{u} \otimes \mathbf{u} + \nabla \mathbf{d} \odot \nabla \mathbf{d} - \frac{1}{2}|\nabla \mathbf{d}|^2 I_3 - \left(F(\mathbf{d})I_3 \right. \right.
$$
$$
\left. \left. - \fint_{\mathbb{P}_2} F(\mathbf{d})I_3 \right) + S_\alpha[\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}), \mathbf{d}] + \fint_{\mathbb{P}_2} S_\alpha[\mathbf{f}(\mathbf{d}), \mathbf{d}] \right] (y) \mathrm{d}y,
$$
(4.85)

and $P^{(2)}(\cdot, t) := (P - P^{(1)})(\cdot, t)$. For $P^{(1)}$, we have that

$$
\nabla^{l-1} P^{(1)}(x) = \int_{\mathbb{R}^3} \nabla_x^2 G(x - y) \nabla^{l-1} \left[\eta \left(\mathbf{u} \otimes \mathbf{u} + \nabla \mathbf{d} \odot \nabla \mathbf{d} - \frac{1}{2}|\nabla \mathbf{d}|^2 I_3 \right. \right.
$$
$$
\left. \left. - (F(\mathbf{d})I_3 - \fint_{\mathbb{P}_2} F(\mathbf{d})I_3) + S_\alpha[\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}), \mathbf{d}] + \fint_{\mathbb{P}_2} S_\alpha[\mathbf{f}(\mathbf{d}), \mathbf{d}]) \right) \right] (y) \mathrm{d}y.
$$

By Calderon–Zygmund's singular integral estimate, with bounds (4.79)–(4.83) we can show that

$$
\int_{\mathbb{P}_{1+2^{-l}}} |\nabla^{l-1} P^{(1)}|^2 \mathrm{d}x \, \mathrm{d}t \leqslant C(l)\varepsilon_1.
$$
(4.86)

We see that $P^{(2)}$ satisfies

$$
-\Delta P^{(2)} = 0 \quad \text{in } B_{1+2^{-(l+1)}+3^{-(l+1)}}.
$$
(4.87)

Then we derive from the regularity of harmonic function that for $1 \leqslant j \leqslant 2l$,

$$
\int_{\mathbb{P}_{1+2^{-(l+1)}+5^{-(l+1)}}} |\nabla^j P^{(2)}|^{\frac{3}{2}} \mathrm{d}x \, \mathrm{d}t \leqslant C \int_{\mathbb{P}_{1+2^{-(l+1)}+4^{-(l+1)}}} |\nabla^{l-1} P^{(2)}|^{\frac{3}{2}} \mathrm{d}x \, \mathrm{d}t
$$

$$
\leqslant C \int_{\mathbb{P}_{1+2^{-l}}} |\nabla^{l-1} P|^{\frac{3}{2}} \mathrm{d}x \, \mathrm{d}t + C \int_{\mathbb{P}_{1+2^{-l}}} |P^{(1)}|^{\frac{3}{2}} \mathrm{d}x \, \mathrm{d}t
$$

$$
\leqslant C(l) \varepsilon_1.
$$

Now take lth order spatial derivative of the equation $(1.2)_1$, we have[2]

$$
\partial_t(\nabla^l \mathbf{u}) + \nabla^l \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) + \nabla^l \nabla P - \nabla^l \Delta \mathbf{u}
$$
$$
= -\nabla^l \nabla \cdot \left[\nabla \mathbf{d} \odot \nabla \mathbf{d} - \frac{1}{2} |\nabla \mathbf{d}| I_3 - F(\mathbf{d}) I_3 + S_\alpha [\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}), \mathbf{d}] \right]. \tag{4.88}
$$

Let $\eta \in C_0^\infty(B_{1+2^{-l}})$. Multiplying (4.88) by $\nabla^l \mathbf{u} \eta^2$ and integrating over B_2, we obtain[3]

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_{B_2} \frac{1}{2} |\nabla^l \mathbf{u}|^2 \eta^2 \, \mathrm{d}x + \int_{B_2} |\nabla^{l+1} \mathbf{u}|^2 \eta^2 \mathrm{d}x
$$

$$
= \int_{B_2} [\nabla^l(\mathbf{u} \otimes \mathbf{u}) : \nabla \nabla^l \mathbf{u} \eta^2 + \nabla^l(\mathbf{u} \otimes \mathbf{u}) : \nabla^l \mathbf{u} \otimes \nabla(\eta^2)] \mathrm{d}x
$$

$$
+ \int_{B_2} \nabla^l P \cdot \nabla^l \mathbf{u} \cdot \nabla(\eta^2) \mathrm{d}x - \int_{B_2} \nabla \nabla^l \mathbf{u} : \nabla^l \mathbf{u} \otimes \nabla(\eta^2) \mathrm{d}x \tag{4.89}
$$

$$
+ \int_{B_2} \nabla^l \left[\nabla \mathbf{d} \odot \nabla \mathbf{d} - \frac{1}{2} |\nabla \mathbf{d}|^2 - F(\mathbf{d}) I_3 - S_\alpha[\mathbf{f}(\mathbf{d}), \mathbf{d}] \right] : \nabla(\nabla^l \mathbf{u} \eta^2) \mathrm{d}x
$$

$$
+ \int_{B_2} \nabla^l S_\alpha[\Delta \mathbf{d}, \mathbf{d}] : (\nabla \nabla^l \mathbf{u} \eta^2 + \nabla^l \mathbf{u} \otimes \nabla(\eta^2)) \mathrm{d}x
$$

$$
:= I_1 + I_2 + I_3 + I_4 + I_5.
$$

Now we have the following estimate:

$$
|I_1| \lesssim \int_{B_2} \left[|\mathbf{u}||\nabla^l \mathbf{u}| + \sum_{j=1}^{l-1} |\nabla^j \mathbf{u}||\nabla^{l-j} \mathbf{u}| \right] (|\nabla^{l+1} \mathbf{u}|\eta^2 + |\nabla^l \mathbf{u}||\eta||\nabla \eta|) \mathrm{d}x
$$

$$
\leqslant \frac{1}{32} \int_{B_2} |\nabla^{l+1} \mathbf{u}|^2 \eta^2 \mathrm{d}x + C \int_{B_2} |\mathbf{u}|^2 |\nabla^l \mathbf{u}|^2 \eta^2 \mathrm{d}x
$$

$$
+ C \int_{B_2} \sum_{j=1}^{l-1} |\nabla^j \mathbf{u}|^2 |\nabla^{l-j} \mathbf{u}|^2 \eta^2 \mathrm{d}x + C \int_{\mathrm{spt}\,\eta} |\nabla^l \mathbf{u}|^2 \mathrm{d}x,
$$

$$
|I_2| \lesssim \int_{B_2} \left[|\nabla^{l-1} P^{(1)}|(|\nabla^{l+1} \mathbf{u}||\eta||\nabla \eta| + |\nabla^l \mathbf{u}||\nabla^2(\eta^2)|) + |\mathbf{u}||\nabla^l(\nabla^l P^{(2)} \nabla \eta^2)|) \right] \mathrm{d}x
$$

---

[2] Strictly speaking, we need to take finite difference quotient $D_h^i \nabla^{l-1}$ of $(1.2)_1$ and then sending $h \to 0$.
[3] Strictly speaking, we need to multiply the equation by $D_h^i \nabla^{l-1} \mathbf{u} \eta^2$.

$$\leqslant \frac{1}{32}\int_{B_2}|\nabla^{l+1}\mathbf{u}|^2\eta^2\mathrm{d}x + C\int_{\mathrm{spt}\,\eta}(|\nabla^{l-1}P^{(1)}|^2 + |\nabla^l\mathbf{u}|^2)\mathrm{d}x$$

$$+ C\int_{\mathrm{spt}\,\eta}(|\mathbf{u}|^3 + |P^{(2)}|^{\frac{3}{2}})\mathrm{d}x$$

$$|I_3| \lesssim \int_{B_2}|\nabla^{l+1}\mathbf{u}||\eta||\nabla^l\mathbf{u}||\nabla\eta|\mathrm{d}x \leqslant \frac{1}{32}\int_{B_2}|\nabla^{l+1}\mathbf{u}|^2\eta^2\,\mathrm{d}x + C\int_{\mathrm{spt}\,\eta}|\nabla^l\mathbf{u}|^2\mathrm{d}x,$$

$$|I_4| \leqslant \int_{B_2}\left(|\nabla^{l+1}\mathbf{d}||\nabla\mathbf{d}| + \sum_{j=1}^{l-1}|\nabla^{j+1}\mathbf{d}||\nabla^{l+1-j}\mathbf{d}| + |\nabla^lF(\mathbf{d})|\right.$$

$$\left.+ |\nabla^l(S_\alpha[f(\mathbf{d}),\mathbf{d}])|\right)(|\nabla^{l+1}\mathbf{u}|\eta^2 + |\nabla^l\mathbf{u}||\nabla(\eta^2)|)\mathrm{d}x$$

$$\leqslant \frac{1}{32}\int_{B_2}|\nabla^{l+1}\mathbf{u}|^2\eta^2\mathrm{d}x + C\int_{B_2}\left(|\nabla^{l+1}\mathbf{d}|^2|\nabla\mathbf{d}|^2\eta^2 + \sum_{j=1}^{l-1}|\nabla^{j+1}\mathbf{d}|^2|\nabla^{l+1-j}\mathbf{d}|^2\eta^2\right)\mathrm{d}x$$

$$+ C\int_{B_2}(|\nabla^lF(\mathbf{d})|^2\eta^2 + |\nabla^lS_\alpha[\mathbf{f}(\mathbf{d}),\mathbf{d}]|^2\eta^2)\mathrm{d}x + C\int_{\mathrm{spt}\,\eta}|\nabla^l\mathbf{u}|^2\mathrm{d}x.$$

For I_5, set $A_\alpha^l := S_\alpha[\nabla^l\Delta\mathbf{d},\mathbf{d}]$, and $B_\alpha^l := \nabla^lS_\alpha[\Delta\mathbf{d},\mathbf{d}] - A_\alpha^l$, then we have

$$I_5 = \int_{B_2}[A_\alpha^l : \nabla\nabla^l\mathbf{u}\eta^2 + B_\alpha^l : \nabla\nabla^l\mathbf{u}\eta^2 + A_\alpha^l : \nabla^l\mathbf{u}\otimes\nabla(\eta^2) + B_\alpha^l : \nabla^l\mathbf{u}\otimes\nabla(\eta^2)]\mathrm{d}x$$

$$=: I_{51} + I_{52} + I_{53} + I_{54}.$$

Then we get

$$|I_{52}| \leqslant \frac{1}{32}\int_{B_2}|\nabla^{l+1}\mathbf{u}|^2\eta^2\mathrm{d}x + C\int_{B_2}|\nabla\mathbf{d}|^2|\nabla^{l+1}\mathbf{d}|^2\eta^2\,\mathrm{d}x$$

$$+ C\int_{B_2}\sum_{j=1}^{l-1}|\nabla^{j+1}\mathbf{d}|^2|\nabla^{l+1-j}\mathbf{d}|^2\eta^2\mathrm{d}x,$$

$$|I_{53}| \leqslant \frac{1}{32}\int_{B_2}|\nabla^{l+2}\mathbf{d}|^2\eta^2\,\mathrm{d}x + C\int_{\mathrm{spt}\,\eta}|\nabla^l\mathbf{u}|^2\mathrm{d}x,$$

$$|I_{54}| \lesssim \int_{\mathrm{spt}\,\eta}|\nabla^l\mathbf{u}|^2\mathrm{d}x + \int_{B_2}|\nabla\mathbf{d}|^2|\nabla^{l+1}\mathbf{d}|^2\eta^2\,\mathrm{d}x + \int_{B_2}\sum_{j=1}^{l-1}|\nabla^{j+1}\mathbf{d}|^2|\nabla^{l+1-j}\mathbf{d}|^2\eta^2\mathrm{d}x.$$

Now we take $(l+1)$th order spatial derivative of the equation $(1.2)_3$, we have

$$\partial_t(\nabla\nabla^l\mathbf{d}) + \nabla\nabla^l(\mathbf{u}\cdot\nabla\mathbf{d}) - \nabla\nabla^lT_\alpha[\nabla\mathbf{u},\mathbf{d}] = \Delta\nabla\nabla^l\mathbf{d} - \nabla\nabla^l\mathbf{f}(\mathbf{d}). \qquad (4.90)$$

Multiplying (4.90) by $\nabla\nabla^l\mathrm{d}\eta^2$ and integrating over B_2, we obtain[4]

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_2}\frac{1}{2}|\nabla^{l+1}\mathbf{d}|^2\eta^2\,\mathrm{d}x + \int_{B_2}|\nabla^{l+2}\mathbf{d}|^2\eta^2\,\mathrm{d}x
$$

$$
= \int_{B_2}\nabla^l(\mathbf{u}\cdot\nabla\mathbf{d})\cdot\nabla\cdot(\nabla\nabla^l\mathbf{d}\eta^2)\mathrm{d}x
$$

$$
- \int_{B_2}[\nabla^l T_\alpha[\nabla\mathbf{u},\mathbf{d}]\cdot\Delta\nabla^l\mathbf{d}\eta^2 + \nabla^l T_\alpha[\nabla\mathbf{u},\mathbf{d}]\cdot(\nabla(\eta^2)\cdot\nabla\nabla^l\mathbf{d})]\mathrm{d}x
$$

$$
- \int_{B_2}\nabla\nabla^l\mathbf{f}(\mathbf{d}):\nabla\nabla^l\mathbf{d}\eta^2\,\mathrm{d}x =: K_1 + K_2 + K_3. \tag{4.91}
$$

Then we have the following estimates:

$$
|K_1| \lesssim \int_{B_2}\left[|\nabla\mathbf{d}||\nabla^l\mathbf{u}| + |\mathbf{u}||\nabla^{l+1}\mathbf{d}| + \sum_{j=1}^{l-1}|\nabla^j\mathbf{u}||\nabla^{l-j+1}\mathbf{d}|\right]\left(|\nabla^{l+2}\mathbf{d}|\eta^2 + |\nabla^{l+1}\mathbf{d}|\eta|\nabla\eta|\right)\mathrm{d}x
$$

$$
\leqslant \frac{1}{32}\int_{B_2}|\nabla^{l+2}\mathbf{d}|^2\eta^2\,\mathrm{d}x + C\int_{B_2}|\nabla\mathbf{d}|^2|\nabla^l\mathbf{u}|^2\eta^2\,\mathrm{d}x + C\int_{B_2}|\mathbf{u}|^2|\nabla^{l+1}\mathbf{d}|^2\eta^2\,\mathrm{d}x
$$

$$
+ C\int_{B_2}\sum_{j=1}^{l-1}|\nabla^j\mathbf{u}|^2|\nabla^{l-j+1}\mathbf{d}|^2\eta^2\,\mathrm{d}x + C\int_{\mathrm{spt}\,\eta}|\nabla^{l+1}\mathbf{d}|^2\mathrm{d}x,
$$

$$
|K_3| \lesssim \int_{B_2}|\nabla^{l+1}\mathbf{d}|^2\eta^2\,\mathrm{d}x + \int_{B_2}|\nabla^{l+1}\mathbf{f}(\mathbf{d})|^2\eta^2\,\mathrm{d}x.
$$

For K_2, we set $C^l_\alpha =: T_\alpha[\nabla\nabla^l\mathbf{u},\mathbf{d}]$, $D^l_\alpha := \nabla^l T_\alpha[\nabla\mathbf{u},\mathbf{d}] - C^l_\alpha$, then we have

$$
K_2 = -\int_{B_2}\left[C^l_\alpha\cdot\Delta\nabla^l\mathbf{d}\eta^2 + D^l_\alpha\cdot\Delta\nabla^l\mathbf{d}\eta^2 + C^l_\alpha\cdot(\nabla(\eta^2)\cdot\nabla\nabla^l\mathbf{d}) + D^l_\alpha\cdot(\nabla(\eta^2)\cdot\nabla\nabla^l\mathbf{d})\right]\mathrm{d}x
$$

$$
=: K_{21} + K_{22} + K_{23} + K_{24}.
$$

Now we estimate

$$
|K_{22}| \leqslant \frac{1}{32}\int_{B_2}|\nabla^{l+2}\mathbf{d}|^2\eta^2\,\mathrm{d}x + C\int_{B_2}|\nabla\mathbf{d}|^2|\nabla^l\mathbf{u}|^2\eta^2\,\mathrm{d}x + C\int_{B_2}\sum_{j=1}^{l-1}|\nabla^j\mathbf{u}|^2|\nabla^{l+1-j}\mathbf{d}|^2\eta^2\,\mathrm{d}x,
$$

$$
|K_{23}| \leqslant \frac{1}{32}\int_{B_2}|\nabla^{l+1}\mathbf{u}|^2\eta^2\,\mathrm{d}x + C\int_{\mathrm{spt}\,\eta}|\nabla^{l+1}\mathbf{d}|^2\mathrm{d}x,
$$

$$
|K_{24}| \lesssim \int_{\mathrm{spt}\,\eta}|\nabla^{l+1}\mathbf{d}|^2\mathrm{d}x + \int_{B_2}|\nabla\mathbf{d}|^2|\nabla^l\mathbf{u}|^2\eta^2\,\mathrm{d}x + \int_{B_2}\sum_{j=1}^{l-1}|\nabla^j\mathbf{u}|^2|\nabla^{l+1-j}\mathbf{d}|^2\eta^2\,\mathrm{d}x.
$$

---

[4] Strictly speaking, we need to multiply the equation by $D^i_h\nabla^l\mathbf{d}\eta^2$.

Combine all estimate above, and with the cancellation $I_{51} = K_{21}$, we arrive at

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_{B_2} \left(|\nabla^l \mathbf{u}|^2 + |\nabla^{l+1}\mathbf{d}|^2 \right) \eta^2 \, \mathrm{d}x + \int_{B_2} \left(|\nabla^{l+1}\mathbf{u}|^2 + |\nabla^{l+2}\mathbf{d}|^2 \right) \eta^2 \, \mathrm{d}x
$$

$$
\leqslant C \int_{B_2} \left(|\mathbf{u}|^2 |\nabla^l \mathbf{u}|^2 \eta^2 + \sum_{j=1}^{l-1} |\nabla^j \mathbf{u}|^2 |\nabla^{l-j}\mathbf{u}|^2 \eta^2 \right) \mathrm{d}x + C \int_{\mathrm{spt}\,\eta} (|\nabla^l \mathbf{u}|^2 + |\nabla^{l+1}\mathbf{d}|^2) \mathrm{d}x
$$

$$
+ C \int_{\mathrm{spt}\,\eta} (|\nabla^l \mathbf{u}|^2 + |\nabla^{l+1}\mathbf{d}|^2 + |\nabla^{l-1}P^{(1)}|^2 + |\mathbf{u}|^3 + |P^{(2)}|^{\frac{3}{2}}) \mathrm{d}x
$$

$$
+ C \int_{B_2} \left(|\nabla \mathbf{d}|^2 |\nabla^{l+1}\mathbf{d}|^2 \eta^2 + \sum_{j=1}^{l-1} |\nabla^{j+1}\mathbf{d}|^2 |\nabla^{l+1-j}\mathbf{d}|^2 \eta^2 \right) \mathrm{d}x
$$

$$
+ C \int_{B_2} (|\nabla^l F(\mathbf{d})|^2 \eta^2 + |\nabla^l S_\alpha[\mathbf{f}(\mathbf{d}), \mathbf{d}]|^2 \eta^2 + |\nabla^{l+1}\mathbf{f}(\mathbf{d})|^2 \eta^2) \mathrm{d}x
$$

$$
+ C \int_{B_2} \left(|\nabla \mathbf{d}|^2 |\nabla^l \mathbf{u}|^2 \eta^2 + |\mathbf{u}|^2 |\nabla^{l+1}\mathbf{d}|^2 \eta^2 + \sum_{j=1}^{l-1} |\nabla^j \mathbf{d}|^2 |\nabla^{l+1-j}\mathbf{d}|^2 \eta^2 \right) \mathrm{d}x
$$

$$
+ C \int_{B_2} \sum_{j=1}^{l-1} |\nabla^j \mathbf{u}|^2 |\nabla^{l+1-j}\mathbf{d}|^2 \eta^2 \, \mathrm{d}x.
$$

$$(4.92)$$

By Sobolev-interpolation inequality, we have

$$
\int_{B_2} |\mathbf{u}|^2 |\nabla^l \mathbf{u}|^2 \eta^2 \, \mathrm{d}x
$$

$$
\leqslant \left\| \nabla^l \mathbf{u}\eta \right\|_{L^6(B_2)} \left\| \nabla^l \mathbf{u}\eta \right\|_{L^2(B_2)} \|\mathbf{u}\|^2_{L^6(\mathrm{spt}\,\eta)}
$$

$$
\leqslant C \left\| \nabla(\nabla^l \mathbf{u}\eta) \right\|_{L^2(B_2)} \left\| \nabla^l \mathbf{u}\eta \right\|_{L^2(B_2)} \|\mathbf{u}\|^2_{L^6(\mathrm{spt}\,\eta)}
$$

$$
\leqslant \frac{1}{32} \int_{B_2} |\nabla^{l+1}\mathbf{u}|^2 \eta^2 \mathrm{d}x + C \int_{\mathrm{spt}\,\eta} |\nabla^l \mathbf{u}|^2 \mathrm{d}x + C\|\mathbf{u}\|^4_{L^6(\mathrm{spt}\,\eta)} \int_{B_2} |\nabla^l \mathbf{u}|^2 \eta^2 \, \mathrm{d}x,
$$

$$
\int_{B_2} |\mathbf{u}|^2 |\nabla^{l+1}\mathbf{d}|^2 \eta^2 \, \mathrm{d}x \leqslant \frac{1}{32} \int_{B_2} |\nabla^{l+2}\mathbf{d}|^2 \eta^2 \mathrm{d}x + C \int_{\mathrm{spt}\,\eta} |\nabla^{l+1}\mathbf{d}|^2 \mathrm{d}x
$$

$$
+ C\|\mathbf{u}\|^4_{L^6(\mathrm{spt}\,\eta)} \int_{B_2} |\nabla^{l+1}\mathbf{d}|^2 \eta^2 \, \mathrm{d}x,
$$

$$
\int_{B_2} |\nabla \mathbf{d}|^2 |\nabla^l \mathbf{u}|^2 \eta^2 \, \mathrm{d}x \leqslant \frac{1}{32} \int_{B_2} |\nabla^{l+1}\mathbf{u}| \eta^2 \mathrm{d}x + C \int_{\mathrm{spt}\,\eta} |\nabla^l \mathbf{u}|^2 \mathrm{d}x
$$

$$
+ C\|\nabla \mathbf{d}\|^4_{L^6(\mathrm{spt}\,\eta)} \int_{B_2} |\nabla^l \mathbf{u}|^2 \eta^2 \, \mathrm{d}x,
$$

$$
\int_{B_2} |\nabla \mathbf{d}|^2 |\nabla^{l+1}\mathbf{d}|^2 \eta^2 \, \mathrm{d}x \leqslant \frac{1}{32} \int_{B_2} |\nabla^{l+2}\mathbf{d}|^2 \eta^2 \mathrm{d}x + C \int_{\mathrm{spt}\,\eta} |\nabla^{l+1}\mathbf{d}|^2 \mathrm{d}x
$$

$$+ C\|\nabla\mathbf{d}\|_{L^6(\text{spt}\,\eta)}^4 \int_{B_2} |\nabla^{l+1}\mathbf{d}|^2 \eta^2 \, \mathrm{d}x.$$

For lower order terms, we have that for $1 \leqslant j \leqslant l-1$,

$$
\begin{aligned}
\int_{B_2} |\nabla^{l-1}\mathbf{u}|^2 |\nabla^j\mathbf{u}|^2 \eta^2 \, \mathrm{d}x &\leqslant \|\nabla^{l-1}\mathbf{u}\eta\|_{L^6(B_2)}^2 \|\nabla^j\mathbf{u}\|_{L^3(\text{spt}\,\eta)}^2 \\
&\leqslant C\|\nabla(\nabla^{l-1}\mathbf{u}\eta)\|_{L^2(B_2)}^2 \|\nabla^j\mathbf{u}\|_{L^3(\text{spt}\,\eta)}^2 \\
&\leqslant C\|\nabla^j\mathbf{u}\|_{L^3(\text{spt}\,\eta)}^2 \int_{B_2} |\nabla^l\mathbf{u}|^2 \eta^2 \, \mathrm{d}x \\
&\quad + C\|\nabla^{l-1}\mathbf{u}\|_{L^3(\text{spt}\,\eta)}^2 \|\nabla^j\mathbf{u}\|_{L^3(\text{spt}\,\eta)}^2,
\end{aligned}
$$

$$
\begin{aligned}
\int_{B_2} |\nabla^l\mathbf{d}|^2 |\nabla^j\mathbf{u}|^2 \eta^2 \, \mathrm{d}x &\leqslant C\|\nabla^j\mathbf{u}\|_{L^3(\text{spt}\,\eta)}^2 \int_{B_2} |\nabla^{l+1}\mathbf{d}|^2 \eta^2 \, \mathrm{d}x \\
&\quad + C\|\nabla^l\mathbf{d}\|_{L^3(\text{spt}\,\eta)}^2 \|\nabla^j\mathbf{u}\|_{L^3(\text{spt}\,\eta)}^2,
\end{aligned}
$$

$$
\begin{aligned}
\int_{B_2} |\nabla^{l-1}\mathbf{u}| |\nabla^{j+1}\mathbf{d}|^2 \eta^2 \, \mathrm{d}x &\leqslant C\|\nabla^{j+1}\mathbf{d}\|_{L^3(\text{spt}\,\eta)}^2 \int_{B_2} |\nabla^l\mathbf{u}|\eta^2 \\
&\quad + C\|\nabla^{l-1}\mathbf{u}\|_{L^3(\text{spt}\,\eta)}^2 \|\nabla^{j+1}\mathbf{d}\|_{L^3(\text{spt}\,\eta)}^2,
\end{aligned}
$$

$$
\begin{aligned}
\int_{B_2} |\nabla^l\mathbf{d}|^2 |\nabla^{j+1}\mathbf{d}|^2 \eta^2 \, \mathrm{d}x &\leqslant C\|\nabla^{j+1}\mathbf{d}\|_{L^3(\text{spt}\,\eta)}^2 \int_{B_2} |\nabla^{l+1}\mathbf{d}|^2 \eta^2 \, \mathrm{d}x \\
&\quad + C\|\nabla^l\mathbf{d}\|_{L^3(\text{spt}\,\eta)}^2 \|\nabla^{j+1}\mathbf{d}\|_{L^3(\text{spt}\,\eta)}^2,
\end{aligned}
$$

and for $1 \leqslant j, k \leqslant l-2$ that

$$\int_{B_2} |\nabla^j\mathbf{u}|^2 |\nabla^{k+1}\mathbf{d}|^2 \eta^2 \, \mathrm{d}x \leqslant C\int_{\text{spt}\,\eta} |\nabla^j\mathbf{u}|^4 \mathrm{d}x + C\int_{\text{spt}\,\eta} |\nabla^{k+1}\mathbf{d}|^4 \mathrm{d}x.$$

Since $|\mathbf{d}| \leqslant M$ in $\mathbb{P}_2$, by the calculus inequality for H^s (cf [14, appendix]), we have for $-4 \leqslant t \leqslant 0$,

$$\|\nabla^l F(\mathbf{d})\|_{L^2(\text{spt}\,\eta)} \lesssim \|\nabla^l\mathbf{d}\|_{L^2(\text{spt}\,\eta)},$$

$$\|\nabla^l S_\alpha[\mathbf{f}(\mathbf{d}), \mathbf{d}]\|_{L^2(\text{spt}\,\eta)} \lesssim \|\nabla^l\mathbf{d}\|_{L^2(\text{spt}\,\eta)},$$

$$\|\nabla^{l+1}\mathbf{f}(\mathbf{d})\|_{L^2(\text{spt}\,\eta)} \lesssim \|\nabla^{l+1}\mathbf{d}\|_{L^2(\text{spt}\,\eta)}.$$

Put all these estimates together, we arrive at

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_{B_2} \left(|\nabla^l \mathbf{u}|^2 + |\nabla^{l+1} \mathbf{d}|^2 \right) \eta^2 \, \mathrm{d}x + \int_{B_2} \left(|\nabla^{l+1} \mathbf{u}|^2 + |\nabla^{l+2} \mathbf{d}|^2 \right) \eta^2 \mathrm{d}x
$$

$$
\leqslant C \int_{\mathrm{spt}\,\eta} \left[|\nabla^l \mathbf{u}|^2 + |\nabla^{l+1} \mathbf{d}|^2 + |\nabla^l \mathbf{d}|^2 + \sum_{j=1}^{l-2} \left(|\nabla^j \mathbf{u}|^4 + |\nabla^{j+1} \mathbf{d}|^4 \right] \mathrm{d}x \right)
$$

$$
+ C \int_{\mathrm{spt}\,\eta} (|\mathbf{u}|^3 + |\nabla^{l-1} P^{(1)}|^2 + |P^{(2)}|^{\frac{3}{2}}) \mathrm{d}x
$$

$$
+ C \left(\left\| \nabla^{l-1} \mathbf{u} \right\|_{L^3(\mathrm{spt}\,\eta)}^4 + \left\| \nabla^l \mathbf{d} \right\|_{L^3(\mathrm{spt}\,\eta)}^4 + \sum_{j=1}^{l-1} \left(\left\| \nabla^j \mathbf{u} \right\|_{L^3(\mathrm{spt}\,\eta)}^4 + \left\| \nabla^{j+1} \mathbf{d} \right\|_{L^3(\mathrm{spt}\,\eta)}^4 \right) \right)
$$

$$
+ C \left(\left\| (\mathbf{u}, \nabla \mathbf{d}) \right\|_{L^6(B_2)}^4 + \sum_{j=1}^{l-1} \left\| (\nabla^j \mathbf{u}, \nabla^{j+1} \mathbf{d}) \right\|_{L^3(B_2)}^2 \right) \times \int_{B_2} (|\nabla^l \mathbf{u}|^2 + |\nabla^{l+1} \mathbf{d}|^2) \eta^2 \mathrm{d}x.
$$

$$
(4.93)
$$

Now let $\eta \in C_0^\infty(B_{1+2^{-(l+1)}+5^{-(l+1)}})$ be a cut-off function of $B_{1+2^{-(l+1)}+10^{-(l+1)}}$. We can apply the Gronwall's inequality to (4.93), together with (4.79)–(4.83) to get

$$
\sup_{-\left(1+2^{-(l+1)}+10^{-(l+1)}\right)^2 \leqslant t \leqslant 0} \int_{B_{1+2^{-(l+1)}+10^{-(l+1)}}} (|\nabla^l \mathbf{u}|^2 + |\nabla^{l+1} \mathbf{d}|^2) \mathrm{d}x
$$

$$
+ \int_{\mathbb{P}_{1+2^{-(l+1)}+10^{-(l+1)}}} \left(|\nabla^{l+1} \mathbf{u}|^2 + |\nabla^{l+2} \mathbf{d}|^2 \right) \mathrm{d}x \, \mathrm{d}t
$$

$$
\leqslant C(l) \varepsilon_1.
$$

$$
(4.94)
$$

Recall that $\nabla^l P$ satisfies

$$
-\Delta \nabla^l P = \mathrm{div}^2 \left[\nabla^l \left(\mathbf{u} \otimes \mathbf{u} + \nabla \mathbf{d} \odot \nabla \mathbf{d} - \frac{1}{2} |\nabla \mathbf{d}|^2 I_3 - (F(\mathbf{d}) I_3 \right. \right.
$$

$$
\left. \left. - \fint_{\mathbb{P}_2} F(\mathbf{d}) I_3 \right) + S_\alpha [\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}), \mathbf{d}] + \fint_{\mathbb{P}_2} S_\alpha [\mathbf{f}(\mathbf{d}), \mathbf{d}] \right) \right].
$$

$$
(4.95)
$$

Then by the Calderón–Zygmund theory and (4.79)–(4.83), (4.94) we can show

$$
\int_{\mathbb{P}_{1+2^{-(l+1)}}} |\nabla^l P|^{\frac{3}{2}} \mathrm{d}x \, \mathrm{d}t \leqslant C(l) \varepsilon_1.
$$

$$
(4.96)
$$

This yields that the conclusion holds for $k = l$. Thus the proof is complete. $\qquad\square$

5. Partial regularity

As a consequence of lemma 4.6, we get the following regularity criteria for (1.2):

Corollary 5.1. *For a suitable weak solution* $(\mathbf{u}, \mathbf{d}, P)$ *to* (1.2)*, if* $z \in \mathbb{R}^3 \times (0, \infty)$ *satisfies*

$$
\begin{cases}
\displaystyle\sup_{0<r<\delta} |\mathbf{d}_{z,r}| < \infty, \\
\displaystyle\liminf_{r\to 0+} \Phi(z, r) = 0,
\end{cases}
\tag{5.1}
$$

Then there exists $\delta_1 > 0$ *such that* $(\mathbf{u}, \mathbf{d}) \in C^\infty(\mathbb{P}_{\delta_1}(z))$.

The following lemma is well-known, see [10].

Lemma 5.2. *Let* $\mathbf{d}$ *be a function in* $L^6(\mathbb{R}^3 \times (0, \infty))$*, and let* $z = (x, t) \in \mathbb{R}^3 \times (0, \infty)$ *such that*

$$
\fint_{\mathbb{P}_r(z)} |\mathbf{d} - \mathbf{d}_{z,r}|^6 dx\, dt \leqslant C r^\delta
\tag{5.2}
$$

for some $\delta > 0$ *and some C depending on* **d** *and z. Then* $\lim_{r\to 0} \mathbf{d}_{z,r}$ *exists, and is finite.*

Next we will control the oscillation of **d**. For $0 < T \leqslant \infty$, denote $Q_T = \mathbb{R}^3 \times (0, T)$. Recall the fractional parabolic Sobolev space $W_p^{1,\frac{1}{2}}(Q_T)$, $1 \leqslant p < \infty$, contains all f's satisfying

$$
\|f\|_{W_p^{1,\frac{1}{2}}(Q_T)} = \|f\|_{L^p(Q_T)} + \|f\|_{\dot{W}_p^{1,\frac{1}{2}}(Q_T)} < \infty,
$$

where

$$
\|f\|_{\dot{W}_p^{1,\frac{1}{2}}(Q_T)} := \left(\int_{Q_T} |\nabla f|^p dt\, dx + \int_{\mathbb{R}^3} \int_0^T \int_0^T \frac{|f(x,t) - f(x,s)|^p}{|t-s|^{1+\frac{p}{2}}} dt\, ds\, dx \right)^{\frac{1}{p}}.
$$

From the global energy estimate (1.6) and the Sobolev embedding theorem, we have

$$
(\mathbf{u}, \nabla\mathbf{d}) \in \left(L_t^\infty L_x^2 \cap L_t^2 H_x^1 \cap L_t^{\frac{10}{3}} L_x^{\frac{10}{3}} \right)(Q_T), \ \mathbf{d} \in L_t^{10} L_x^{10}(Q_T).
\tag{5.3}
$$

It follows that

$$
\partial_t \mathbf{d} = \Delta\mathbf{d} - \mathbf{f}(\mathbf{d}) - \mathbf{u} \cdot \nabla\mathbf{d} + T_\alpha[\nabla\mathbf{u}, \mathbf{d}] \in L^{\frac{5}{3}}(Q_T).
$$

From the fractional Galiardo–Nirenberg inequality [1, 2], we get $\mathbf{d} \in W_{\frac{20}{7}}^{1,\frac{1}{2}}(Q_T)$, and

$$
\|\mathbf{d}\|^2_{W_{\frac{20}{7}}^{1,\frac{1}{2}}(Q_T)} \leqslant C\|\mathbf{d}\|_{L^{10}(Q_T)} \|(\partial_t\mathbf{d}, \nabla\mathbf{d})\|_{L^{\frac{5}{3}}(Q_T)} + C\|\mathbf{d}\|^2_{L_t^{\frac{20}{7}} W_x^{1,\frac{20}{7}}(Q_T)} < \infty.
$$

Then the parabolic Sobolev–Poincaré inequality yields

$$
\left(\fint_{\mathbb{P}_r(z)} |\mathbf{d} - \mathbf{d}_{z,r}|^p dx\, dt \right)^{\frac{1}{p}} \leqslant C \left[r^{\frac{20}{7}-5} \int_{\mathbb{P}_r(z)} |\nabla\mathbf{d}|^{\frac{20}{7}} \right.
$$

$$
\left. + r^{\frac{20}{7}-5} \int_{B_r(x)} \int_{t-r^2}^t \int_{t-r^2}^t \frac{|\mathbf{d}(x,s_1) - \mathbf{d}(x,s_2)|^{\frac{20}{7}}}{|s_1 - s_2|^{1+\frac{10}{7}}} ds_1\, ds_2\, dx \right]^{\frac{7}{20}}.
$$

where $p = \frac{5 \cdot \frac{20}{7}}{5 - \frac{20}{7}} = \frac{20}{3} > 6$. Hence by Hölder inequality we have that

$$
\left(\fint_{\mathbb{P}_r(z)} |\mathbf{d} - \mathbf{d}_{z,r}|^6 \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{6}} \leqslant \left(\fint_{\mathbb{P}_r(z)} |\mathbf{d} - \mathbf{d}_{z,r}|^{\frac{20}{3}} \mathrm{d}x \, \mathrm{d}t \right)^{\frac{3}{20}}
$$

$$
\leqslant C \left[r^{\frac{20}{7} - 5} \int_{\mathbb{P}_r(z)} |\nabla \mathbf{d}|^{\frac{20}{7}} \right. \tag{5.4}
$$

$$
\left. + r^{\frac{20}{7} - 5} \int_{B_r(x)} \int_{t-r^2}^t \int_{t-r^2}^t \frac{|\mathbf{d}(x, s_1) - \mathbf{d}(x, s_2)|^{\frac{20}{7}}}{|s_1 - s_2|^{1 + \frac{10}{7}}} \mathrm{d}s_1 \, \mathrm{d}s_2 \, \mathrm{d}x \right]^{\frac{7}{20}}.
$$

Proof of theorem 1. Define

$$
\Sigma = \left\{ z \in \mathbb{R}^3 \times (0, \infty) : \liminf_{r \to 0} \Phi(z, r) > \varepsilon_2^6 \text{ or } \liminf_{r \to 0} |\mathbf{d}_{z,r}| = \infty \right\}.
$$

It follows from corollary 5.1 that Σ is closed and $(\mathbf{u}, \mathbf{d}) \in C^\infty(\mathbb{R}^3 \times (0, \infty) \setminus \Sigma)$. From (5.4) and lemma 5.2, we know that $\Sigma \subset \cap_{\sigma > 0} \mathcal{S}_\sigma$, where $\mathcal{S}_\sigma$ is defined by

$$
\mathcal{S}_\sigma = \left\{ z \in Q_T : \liminf_{r \to 0} \left[r^{-\frac{5}{3}} \int_{\mathbb{P}_r(z)} \left(|\mathbf{u}|^{\frac{10}{3}} + |\nabla \mathbf{d}|^{\frac{10}{3}} \right) \mathrm{d}x \, \mathrm{d}t \right. \right.
$$

$$
\left. + \left(r^{-\frac{5}{3}} \int_{\mathbb{P}_r(z)} |P|^{\frac{5}{3}} \mathrm{d}x \, \mathrm{d}t \right)^2 \right] > 0, \text{ or}
$$

$$
\liminf_{r \to 0} r^{-\frac{15}{7} - \sigma} \left(\int_{\mathbb{P}_r(z)} |\nabla \mathbf{d}|^{\frac{20}{7}} \mathrm{d}x \, \mathrm{d}t + \int_{B_r(x)} \int_{t-r^2}^t \int_{t-r^2}^t \right.
$$

$$
\left. \left. \times \frac{|\mathbf{d}(x, s_1) - \mathbf{d}(x, s_2)|^{\frac{20}{7}}}{|s_1 - s_2|^{1 + \frac{10}{7}}} \mathrm{d}s_1 \, \mathrm{d}s_2 \, \mathrm{d}x \right) > 0 \right\}.
$$

For the last integral, we have that

$$
f(x, s_1, s_2) = \frac{|\mathbf{d}(x, s_1) - \mathbf{d}(x, s_2)|^{\frac{20}{7}}}{|s_1 - s_2|^{1 + \frac{10}{7}}} \in L^1(\mathbb{R}^3 \times (0, T) \times (0, T)).
$$

Let $\tilde{\delta}$ be the metric on $\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$:

$$
\tilde{\delta}(\xi_1, \xi_2) = \max \left\{ |x_1 - x_2|, \sqrt{|t_1 - t_2|}, \sqrt{|s_1 - s_2|} \right\}, \ \forall \, \xi_i = (x_i, t_i, s_i) \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}.
$$

A standard covering argument implies that

$$
\widetilde{\mathcal{P}}^{\frac{15}{7} + \sigma} \left\{ (x, s, t) \in \mathbb{R}^3 \times (0, T) \times (0, T) : \liminf_{r \to 0+} r^{-\frac{15}{7} - \sigma} \int_{B_r(x)} \int_{s-r^2}^s \int_{t-r^2}^t f(\xi) \mathrm{d}\xi > 0 \right\} = 0,
$$

where $\widetilde{\mathcal{P}}^k$ denotes the k-dimensional Hausdorff measure on $\mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R}_+$ with respect to the metric $\tilde{\delta}$.

Since the map $T(x,t) = (x,t,t) : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$ is an isometric embedding of $(\mathbb{R}^3 \times \mathbb{R}, \delta)$ into $(\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}, \tilde{\delta})$, we have that

$$
\begin{aligned}
&\mathcal{P}^{\frac{15}{7}+\sigma} \left(\left\{ (x,t) \in Q_T : \liminf_{r\to 0+} r^{-\frac{15}{7}-\sigma} \int_{B_r(x)} \int_{t-r^2}^t \int_{t-r^2}^t f(\xi) \mathrm{d}\xi > 0 \right\} \right) \\
&= \widetilde{\mathcal{P}}^{\frac{15}{7}+\sigma} \left(T \left[\left\{ (x,t) \in Q_T : \liminf_{r\to 0+} r^{-\frac{15}{7}-\sigma} \int_{B_r(x)} \int_{t-r^2}^t \int_{t-r^2}^t f(\xi) \mathrm{d}\xi > 0 \right\} \right] \right) \\
&= \widetilde{\mathcal{P}}^{\frac{15}{7}+\sigma} \left(\left\{ (x,t,t) \in Q_T \times (0,T) : \liminf_{r\to 0+} r^{-\frac{15}{7}-\sigma} \int_{B_r(x)} \int_{t-r^2}^t \int_{t-r^2}^t f(\xi) \mathrm{d}\xi > 0 \right\} \right) \\
&\leqslant \widetilde{\mathcal{P}}^{\frac{15}{7}+\sigma} \left(\left\{ (x,s,t) \in Q_T \times (0,T) : \liminf_{r\to 0+} r^{-\frac{15}{7}-\sigma} \int_{B_r(x)} \int_{s-r^2}^s \int_{t-r^2}^t f(\xi) \mathrm{d}\xi > 0 \right\} \right) \\
&= 0.
\end{aligned}
$$

(5.5)

Again, by a simple covering argument we can show

$$
\mathcal{P}^{\frac{15}{7}+\sigma} \left(\left\{ z \in Q_T : r^{-\frac{15}{7}-\sigma} \int_{\mathbb{P}_r(z)} |\nabla \mathbf{d}|^{\frac{20}{7}} \mathrm{d}x \, \mathrm{d}t > 0 \right\} \right) = 0,
$$

(5.6)

and

$$
\begin{aligned}
\mathcal{P}^{\frac{5}{3}} \bigg(\bigg\{ z \in Q_T : \lim_{r\to 0} r^{-\frac{5}{3}} \int_{\mathbb{P}_r(z)} \left(|\mathbf{u}|^{\frac{10}{3}} + |\nabla \mathbf{d}|^{\frac{10}{3}} \right) \mathrm{d}x \, \mathrm{d}t \\
+ \left(r^{-\frac{5}{3}} \int_{\mathbb{P}_r(z)} |P|^{\frac{5}{3}} \right)^2 > 0 \bigg\} \bigg) = 0.
\end{aligned}
$$

(5.7)

It follows from (5.5), (5.6) and (5.7) that $\mathcal{P}^{\frac{15}{7}+\sigma}(\mathcal{S}_\sigma) = 0$ so that $\mathcal{P}^{\frac{15}{7}+\sigma}(\Sigma) = 0$, $\forall \sigma > 0$. $\square$

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ORCID iDs

Hengrong Du ⓘ https://orcid.org/0000-0003-2392-8963

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