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On a thermodynamically consistent model for magnetoviscoelastic fluids in 3D

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Abstract. We introduce a system of equations that models a non-isothermal magnetoviscoelastic fluid. We show that the model is thermodynamically consistent, and that the critical points of the entropy functional with prescribed energy correspond exactly with the equilibria of the system. The system is investigated in the framework of quasilinear parabolic systems and shown to be locally well-posed in an L_p-setting. Furthermore, we prove that constant equilibria are normally stable. In particular, we show that solutions that start close to a constant equilibrium exist globally and converge exponentially fast to a (possibly different) constant equilibrium. Finally, we establish that the negative entropy serves as a strict Lyapunov functional and we then show that every solution that is eventually bounded in the topology of the natural state space exists globally and converges to the set of equilibria.

1. Introduction

We study the following system of equations that models the evolution of a magnetoviscoelastic fluid, allowing for a non-constant temperature in a C^3-bounded domain $\Omega \subset \mathbb{R}^3$ with outward unit normal ν:

$$\begin{cases} \partial_t u + u\cdot\nabla u - \nabla\cdot(\mu(\theta)\nabla u) + \nabla\pi = -\nabla\cdot(\nabla m \odot \nabla m) + \nabla\cdot(FF^{\mathsf{T}}) & \text{in} \quad \Omega, \\ \nabla\cdot u = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ \partial_t F + u\cdot\nabla F - \nabla\cdot(\kappa(\theta)\nabla F) = (\nabla u)^{\mathsf{T}} F & \text{in} \quad \Omega, \\ F = 0 & \text{on} \quad \partial\Omega, \\ \partial_t\theta + u\cdot\nabla\theta + \nabla\cdot q = \mu(\theta)|\nabla u|^2 + \kappa(\theta)|\nabla F|^2 + \alpha(\theta)|\Delta m + |\nabla m|^2 m|^2 & \text{in} \quad \Omega, \\ q\cdot\nu = 0 & \text{on} \quad \partial\Omega, \\ \partial_t m + u\cdot\nabla m = -\alpha(\theta) m\times(m\times\Delta m) - \beta(\theta) m\times\Delta m & \text{in} \quad \Omega, \\ \partial_\nu m = 0 & \text{on} \quad \partial\Omega, \\ |m| = 1 & \text{in} \quad \Omega, \\ (u(0), F(0), \theta(0), m(0)) = (u_0, F_0, \theta_0, m_0) & \text{in} \quad \Omega. \end{cases} \tag{1.1}$$

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The meaning of the expressions $\nabla m \odot \nabla m$, $\nabla \cdot (\nabla m \odot \nabla m)$, ∇u, and $\nabla \cdot F$ is explained below in the paragraph titled notation. The equations above comprise a coupled system, consisting of

1. the incompressible Navier–Stokes equations with variable viscosity coefficient $\mu(\theta)$ for the velocity field

$$u : (0, T) \times \Omega \to \mathbb{R}^3$$

with a right-hand side that includes the elastic stress tensor induced by the magnetization field m and the deformation tensor F. Moreover, $\pi : (0, T) \times \Omega \to \mathbb{R}$ denotes the pressure function;
2. a transport-dissipative system for the deformation tensor

$$F : (0, T) \times \Omega \to \mathbb{M}^3 := \mathbb{R}^{3\times 3}$$

with variable dissipative coefficient $\kappa(\theta)$ and stretching term $(\nabla u)^\top F$;
3. a transported anisotropic heat equation for the (absolute) temperature function

$$\theta : (0, T) \times \Omega \to \mathbb{R}$$

with the heat flux q given by the generalized Fourier law [9,36]

$$q = q(u, F, \theta, m) = -K(u, F, \theta, m)\nabla\theta, \tag{1.2}$$

where $K(u, F, \theta, m)$ is a positive-definite, matrix-valued function of (u, F, θ, m) which reflects the inhomogeneity of the material. For example, one may choose

$$q = -h(\theta)\nabla\theta - k(\theta)(\nabla\theta \cdot m)m,$$

where $h(\theta)$ describes the variable heat conductivity, while $k(\theta)$ represents the inhomogeneous thermal conductivity along the direction preferred by the magnetization within the medium. In this case, $K(u, F, \theta, m) = h(\theta)I_3 + k(\theta)(m \otimes m)$;
4. a convected Landau–Lifshitz–Gilbert system for the magnetization field

$$m : (0, T) \times \Omega \to \mathbb{S}^2 = \{d \in \mathbb{R}^3 : |d| = 1\}$$

with variable Gilbert damping parameter $\alpha(\theta)$ and exchange parameter $\beta(\theta)$.

The model can be used to describe so-called smart fluids (magnetorheological fluids), that is, fluids carrying magnetoelastic particles. Because of their remarkable properties, magnetoelastic materials are widely used in technical applications.

When fluids are subjected to heat, their molecules experience internal movement due to changes in temperature. This compound effect can be described by differential equations that govern the laws of these changes. While numerous publications have been devoted to the dynamics of magnetoviscoelastic fluids in the isothermal case (see, for example, [5,7,10,14,15,21,34,38]), there is a lack of research on the thermodynamic effects associated with these fluids.

A related class of thermodynamically consistent models for incompressible non-isothermal *nematic liquid crystal flows* has been developed using the Ericksen–Leslie formalism, as discussed in [6,11,17–19]. Specifically, De Anna and Liu extended the general Ericksen–Leslie system and the general Oseen–Frank energy density in [6], and they obtain a global well-posedness result of strong solutions for initial data that are close to equilibrium in suitable homogeneous Besov spaces. Another approach to modeling non-isothermal nematic liquid crystals is presented in [11], where the authors introduce an energetically closed system and derive the equations using a generalized variational principle. They show the existence of global weak solutions for suitable initial data in a three-dimensional bounded domain with a sufficiently regular boundary. Meanwhile, Hieber and Prüss analyzed the non-isothermal Ericksen–Leslie system by means of maximal L_p-regularity techniques for quasilinear parabolic evolution equations in [17,19]. They demonstrate the local existence of classical solutions and the stability of solutions subject to initial data that are close enough to an equilibrium for the case of linear boundary conditions. It is also worth noting that the authors in [12,13] consider non-isothermal nematics using the Q-tensor as the order parameter. They prove global existence of weak solutions with the Landau–De Gennes polynomial potential and the Ball–Majumdar singular potential, which are commonly used to describe the configuration of the liquid crystal molecules.

A detailed analysis reveals that system (1.1) features a quasilinear parabolic structure, and we are employing the theory of maximal regularity, see for instance [32], to study existence, uniqueness, and qualitative properties of solutions. With this approach in mind, there are several difficulties that arise in the mathematical analysis. For instance,

- The equation $(1.1)_6$ for the temperature θ contains terms that have a highly nonlinear dependence on the magnetic field m; namely, the equation contains a term that is quadratic in second-order derivatives of m. On a technical level, this means that we need to work in function spaces that encode higher regularity for the magnetic field m.
- The flux vector q in (1.2) is allowed to depend in a nonlinear way on the variables (u, F, θ, m). The boundary condition $q \cdot \nu = 0$ then leads to a nonlinear boundary condition which is to be satisfied by the solutions. This implies that solutions 'live on a nonlinear manifold,' and this adds significant challenges to a mathematical treatment.

Quasilinear parabolic systems with nonlinear boundary conditions have been studied in the literature by several authors, for instance in [20,22,23,25–27]. However, the results contained in these publications cannot be applied directly to the model (1.1). For instance, while these works cover a very general class of quasilinear parabolic systems with nonlinear boundary conditions, they do not include a coupling to the Navier–Stokes equations. Moreover, these works do not feature equations which contain quadratic terms in the highest derivatives of some of the variables.

Due to the structure of the nonlinear terms in the temperature equation, it seems infeasible to find a realization of (1.1) in so-called extrapolation spaces, which would be helpful in order to absorb nonlinear boundary conditions of co-normal type, see for instance [2,35].

Finally, we would like to mention that the techniques developed in this paper can be generalized to also cover fully nonlinear boundary conditions, and we could also admit nonlinear boundary conditions for the remaining variables.

The manuscript is structured as follows. In Sect. 2, we show that system (1.1) is thermodynamically consistent. In addition, we provide a characterization of the equilibria and we show that the critical points of the constrained entropy functional correspond exactly to these equilibria. In Sect. 3, we introduce a functional analytic setting to study the system (1.1). In Sect. 4, we provide existence and uniqueness results for some related linear problems, which will then form the basis to establish the local well-posedness of strong solutions for system (1.1), carried out in Sect. 5. In the main theorem of this section, Theorem 5.3, we show that system (1.1) generates a Lipschitz continuous semiflow on the state manifold (defined by the nonlinear boundary condition). Here, we have been inspired by the approach in [22,26,27]. In addition, we show that the temperature satisfies a maximum principle. In Sect. 6, we provide criteria for global existence. In addition, we study stability of constant equilibria; in particular, we show that solutions that start close to a constant equilibrium exist globally and converge exponentially fast to a (possibly different) constant equilibrium. Finally, in “Appendix A,” we establish some relevant properties of fractional Sobolev spaces with temporal weights, and in “Appendix B,” we study mapping properties of the nonlinearities associated with system (1.1).

Notation: For the readers’ convenience, we list here some notations and conventions used throughout the manuscript.

In the following, all vectors $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ are viewed as column vectors. For two vectors $a, b \in \mathbb{R}^n$, the Euclidean inner product is denoted by $a \cdot b$. Given two matrices $A, B \in \mathbb{M}^n$, the Frobenius matrix inner product $A : B$ is given by

$$A : B = \operatorname{trace}(AB^{\mathsf{T}}),$$

where $^{\mathsf{T}}$ is the transpose. Suppose Ω is an open subset of $\mathbb{R}^n$. If $u \in C^1(\Omega; \mathbb{R}^n)$, we set $\nabla u(x) = e_j \otimes \partial_j u(x)$ for $x \in \Omega$. Hence, for $u = (u_1, \dots, u_n) \in C^1(\Omega; \mathbb{R}^n)$, we have

$$[\nabla u(x)]_{ij} = \partial_i u_j(x), \quad 1 \le i, j \le n, \quad x \in \Omega.$$

We note that $[\nabla u(x)]^{\mathsf{T}}$ corresponds to the Fréchet derivative of u at $x \in \Omega$.

If $A \in C^1(\Omega; \mathbb{M}^n)$, its divergence $\nabla \cdot A$ is the vector function defined by

$$(\nabla \cdot A)(x) = (\partial_j A(x))^{\mathsf{T}} e_j, \quad x \in \Omega. \tag{1.3}$$

Hence, if $A = [a_{ij}] \in C^1(\Omega; \mathbb{M}^n)$, its divergence is given by

$$[(\nabla \cdot A)(x)]_i = \partial_j a_{ji}(x), \quad i = 1, \dots, n, \quad x \in \Omega.$$

Here and in the sequel, we use the summation convention, indicating that terms with repeated indices are added. We note that (1.3) implies

$$(\nabla \cdot A) \cdot u = \nabla \cdot (Au) - A : \nabla u, \quad A \in C^1(\Omega; \mathbb{M}^n),\ u \in C^1(\Omega; \mathbb{R}^n). \tag{1.4}$$

For a matrix $A \in C^1(\Omega; \mathbb{M}^n)$, we set $|\nabla A|^2 = \partial_j A : \partial_j A$. Finally, for $m \in C^1(\Omega; \mathbb{R}^3)$, $\nabla m \odot \nabla m$ denotes the symmetric tensor given by $[\nabla m \odot \nabla m]_{ij} = (\partial_i m | \partial_j m)$, $1 \le i, j \le 3$.

For functions $f, g \in L_2(\Omega; \mathbb{R}^m)$,

$$(f|g)_\Omega = \int_\Omega f \cdot g \,\mathrm{d}x$$

denotes the L_2-inner product. For any Banach space X, $s \ge 0$, $p \in (1, \infty)$, $W^s_p(\Omega; X)$ denote the X-valued Sobolev(-Slobodeckij) spaces. When the choice of X is clear from the context, we will just write $W^s_p(\Omega)$.

Given any $T \in (0, \infty]$, we will denote the interval $(0, T)$ by J_T. For $p \in (1, \infty)$ and $\mu \in (0, 1]$, the X-valued L_p-spaces with temporal weight are defined by

$$L_{p,\mu}(J_T; X) := \left\{ f : (0, T) \to X : [t \mapsto t^{1-\mu} f(t)] \in L_p(J_T; X) \right\}.$$

Similarly, for $k \in \mathbb{N}$,

$$W^k_{p,\mu}(J_T; X) := \left\{ f \in W^k_{1,loc}(J_T; X) : \partial_t^j f \in L_{p,\mu}(J_T; X),\ j = 0, 1, \ldots, k \right\}.$$

For $s \in (0, 1)$, the Sobolev–Slobodeckij spaces with temporal weights are defined as

$$W^s_{p,\mu}(J_T; X) := \{u \in L_{p,\mu}(J_T; X) : \|u\|_{W^s_{p,\mu}(J_T;X)} < \infty\},$$

where

$$\begin{aligned} \|u\|_{W^s_{p,\mu}(J_T;X)} &= \|u\|_{L_{p,\mu}(J_T;X)} + [u]_{W^s_{p,\mu}(J_T;X)}, \\ [u]_{W^s_{p,\mu}(J_T;X)} &:= \left(\int_0^T \int_0^t \tau^{p(1-\mu)} \frac{\|u(t) - u(\tau)\|^p_X}{(t-\tau)^{sp+1}} \, d\tau dt \right)^{1/p}, \end{aligned} \tag{1.5}$$

see [28, Formula (2.6)]. $\|\cdot\|_{W^s_{p,\mu}(J_T;X)}$ is termed the intrinsic norm of $W^s_{p,\mu}(J_T; X)$.

For any two Banach spaces X and Y, the notation $\mathcal{L}(X, Y)$ stands for the set of all bounded linear operators from X to Y and $\mathcal{L}(X) := \mathcal{L}(X, X)$. $\mathcal{L}is(X, Y)$ denotes the subset of $\mathcal{L}(X, Y)$ consisting of linear isomorphisms from X to Y.

Finally, in this article, $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ always denotes a continuous non-decreasing function satisfying

$$\Phi(r) \to 0^+ \quad \text{as } r \to 0^+.$$

2. Thermodynamic consistency

In this section, we discuss the thermodynamic properties of (1.1). We introduce the following assumptions:

$$\begin{aligned}&\mu, \kappa, \alpha, \beta \in C^5(\mathbb{R}), \quad K \in C^5(\mathbb{R}^3 \times \mathbb{M}^3 \times \mathbb{R} \times \mathbb{R}^3; \mathrm{sym}\,(\mathbb{M}^3));\\&\mu \geq \underline{\mu}, \quad \kappa \geq \underline{\kappa}, \quad \alpha \geq \underline{\alpha}, \quad K \geq \underline{c} I_3,\end{aligned} \tag{2.1}$$

where $\underline{\mu}$, $\underline{\kappa}$, $\underline{\alpha}$ and $\underline{c}$ are given positive constants. Here, we assume C^5-smoothness for convenience.

We assume that the Helmholtz free energy density ψ is given by

$$\psi = \psi(F, \theta, m) = \frac{1}{2}|F|^2 + \frac{1}{2}|\nabla m|^2 - \theta \ln \theta.$$

Then, the entropy density η and the internal energy density e_{int} can be obtained via the following thermodynamic relations, see for instance [32, page 7]:

$$\begin{aligned}&\eta = -\partial_\theta \psi = 1 + \ln \theta && \text{(the Maxwell relation)}\\&e_{\mathrm{int}} = \psi + \theta\eta = \tfrac{1}{2}|F|^2 + \tfrac{1}{2}|\nabla m|^2 + \theta && \text{(the Legendre transform of } \psi \text{ w. r. t. } \eta).\end{aligned}$$

We can derive from the equation for θ in (1.1) the entropy evolution

$$\partial_t \eta + u \cdot \nabla \eta + \nabla \cdot g = r, \tag{2.2}$$

where g denotes the entropy flux which satisfies the Clausius–Duhem relation (see for instance page 7 in [32])

$$g = \frac{q}{\theta},$$

and where the entropy production rate r is given as

$$r = \frac{1}{\theta}\left[\mu(\theta)|\nabla u|^2 + \kappa(\theta)|\nabla F|^2 + \alpha(\theta)|\Delta m + |\nabla m|^2 m|^2 - \frac{q \cdot \nabla \theta}{\theta}\right]. \tag{2.3}$$

The *thermodynamic consistency* of (1.1) is established in the following result.

Proposition 2.1. *Suppose* (u, F, θ, m) *is a solution of* (1.1) *with the regularity properties asserted in Theorem* 5.3. *Then, the following properties hold.*

(a) (*First law of thermodynamics*). *The total energy*

$$\mathsf{E} = \mathsf{E}(u, F, \theta, m) = \int_\Omega (\frac{1}{2}|u|^2 + e_{\mathrm{int}})\,\mathrm{d}x = \int_\Omega (\frac{1}{2}|u|^2 + \frac{1}{2}|F|^2 + \frac{1}{2}|\nabla m|^2 + \theta)\,\mathrm{d}x$$

is preserved along the solution (u, F, θ, m).

(b) (*Second law of thermodynamics*). *The total entropy*

$$\mathsf{N} = \mathsf{N}(\theta) = \int_\Omega \eta\,\mathrm{d}x = \int_\Omega (1 + \ln \theta)\,\mathrm{d}x$$

is non-decreasing along the solution (u, F, θ, m). *In fact, the entropy production rate* r *is always non-negative, i.e.,* $r \geq 0$.

Proof. Let (u, F, θ, m) be a solution of (1.1) with initial value $z_0 = (u_0, F_0, \theta_0, m_0)$ defined on its maximal interval of existence $[0, T_+(z_0))$, see Theorem 5.3 for the precise regularity assertions. Let $T \in (0, T_+(z_0))$ be fixed. For notational simplicity, we suppress the time variable in the following computations.

For (a), we follow the calculations in [10, Proposition 4.1] to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\int_\Omega \frac{1}{2}\left(|u|^2 + |F|^2 + |\nabla m|^2\right)\mathrm{d}x\\&= -\int_\Omega \left[\mu(\theta)|\nabla u|^2 + \kappa(\theta)|\nabla F|^2 + \alpha(\theta)|\Delta m + |\nabla m|^2 m|^2\right]\mathrm{d}x, \quad t \in (0, T).\end{aligned} \tag{2.4}$$

Meanwhile, integrating the equation for θ in (1.1) over Ω, from an integration by parts and the relations $\nabla \cdot u = 0$ and $(u, q \cdot \nu) = (0, 0)$ on $\partial\Omega$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \theta\,\mathrm{d}x = \int_\Omega \left[\mu(\theta)|\nabla u|^2 + \kappa(\theta)|\nabla F|^2 + \alpha(\theta)|\Delta m + |\nabla m|^2 m|^2\right]\mathrm{d}x, \quad t \in (0, T). \tag{2.5}$$

Now, adding (2.4) and (2.5) yields $\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{E} = 0$ for $t \in (0, T)$.

For (b), we obtain from the equation for θ in (1.1), the Maxwell relation $\eta = 1 + \ln\theta$, and (2.2) that

$$\begin{aligned}&\partial_t \eta + u \cdot \nabla\eta\\&= \frac{1}{\theta}(\partial_t\theta + u \cdot \nabla\theta)\\&= \frac{1}{\theta}\left(-\nabla \cdot q + \mu(\theta)|\nabla u|^2 + \kappa(\theta)|\nabla F|^2 + \alpha(\theta)|\Delta m + |\nabla m|^2 m|^2\right)\\&= -\nabla \cdot g + \frac{1}{\theta}\left[\mu(\theta)|\nabla u|^2 + \kappa(\theta)|\nabla F|^2 + \alpha(\theta)|\Delta m + |\nabla m|^2 m|^2 - \frac{q \cdot \nabla\theta}{\theta}\right]\\&= -\nabla \cdot g + r, \qquad t \in (0, T).\end{aligned}$$

By (1.2), we have $-q \cdot \nabla\theta = K\nabla\theta \cdot \nabla\theta \geq \underline{c}|\nabla\theta|^2 \geq 0$, and hence $r \geq 0$. This implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{N} = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta\,\mathrm{d}x = \int_\Omega (-u \cdot \nabla\eta - \nabla \cdot g + r)\,\mathrm{d}x = \int_\Omega r\,\mathrm{d}x \geq 0, \quad t \in (0, T), \tag{2.6}$$

and hence the assertion holds. □

2.1. Entropy and equilibria

Here, we follow the arguments in [32, Section 1.2], see also [19], to discuss the equilibria of system (1.1) and their connection to the critical points of the entropy functional. We begin with a characterization of the equilibria.

Proposition 2.2. (a) *The set $\mathcal{E}$ of equilibria of* (1.1) *is given by*

$$\mathcal{E} = \{(u_*, F_*, \theta_*, m_*) \in \{0\} \times \{0\} \times (0, \infty) \times C^\infty(\overline{\Omega})\},$$

where m_* satisfies the harmonic map equation with homogeneous Neumann boundary condition

$$\begin{cases} \Delta m + |\nabla m|^2 m = 0 & \text{in} \quad \Omega, \\ |m| \equiv 1 & \text{in} \quad \Omega, \\ \partial_\nu m = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.7}$$

Moreover, the equilibrium pressure is given by $\pi_* = -\frac{1}{2}|\nabla m_*|^2 + C$, where C is some constant.

(b) $-\mathsf{N}$ *is a strict Lyapunov function for* (1.1).

Proof. (a) Suppose $z_* = (u_*, F_*, m_*, \theta_*)$ is an equilibrium for (1.1). Then, $\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{N} = 0$ and it follows from (2.3), (2.6) that $(\nabla u_*, \nabla F_*, \nabla\theta_*, \Delta m_* + |\nabla m_*|^2 m_*) = (0, 0, 0, 0)$, as

$$-q_* \cdot \nabla\theta_* = K(z_*)\nabla\theta_* \geq \underline{c}|\nabla\theta_*|^2.$$

The boundary conditions $(u_*, F_*) = (0, 0)$ readily imply $(u_*, F_*) = (0, 0)$. In addition, we conclude that θ_* is a constant. Finally, we can derive from $(1.1)_1$ that $\nabla\pi_* = -\nabla(\frac{1}{2}|\nabla m_*|^2)$, and hence $\pi_* = -(\frac{1}{2}|\nabla m_*|^2 + C)$ with some constant C.

(b) Let (u, F, θ, m) be a solution of (1.1) with the regularity properties of Theorem 5.3, defined on the maximal interval of existence $[0, T_+(z_0))$. Suppose that $\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{N} = 0$ on some interval $(t_1, t_2) \subset (0, T_+(z_0))$. Then, $r \geq 0$ implies $r(t) \equiv 0$ in Ω for $t \in (t_1, t_2)$. We conclude as above that $(u(t), F(t)) = (0, 0)$ and $\theta(t) = c$ for $t \in (t_1, t_2)$, where $c > 0$ is a constant. Therefore,

$$(\partial_t u(t), \partial_t F(t), \partial_t \theta(t)) = (0, 0, 0), \quad t \in (t_1, t_2).$$

Moreover, $\Delta m(t) + |\nabla m(t)|^2 m(t) = 0$ for $t \in (t_1, t_2)$. Taking the cross-product of both sides of this equation with $m(t)$ results in

$$m(t) \times \Delta m(t) = -|\nabla m(t)|^2 (m(t) \times m(t)) = 0, \quad t \in (t_1, t_2).$$

Hence, we conclude that $\partial_t m(t) = 0$ for $t \in (t_1, t_2)$. This shows that the solution is at equilibrium. □

We will now provide an informal discussion concerning the critical points of the *constrained entropy functional.*

Suppose that (u, F, θ, m) is a sufficiently smooth critical point of N with $\theta > 0$, subject to the constraints $G(m) = (|m|^2 - 1)/2 = 0$, $\mathsf{E} = \mathsf{E}_0$, and the boundary conditions in (1.1). By the Lagrange multiplier method, we get $\lambda_{\mathsf{E}} \in \mathbb{R}$ and $\lambda_G \in$

$L_2(\Omega)$ such that the first variation of N at $z = (u, F, \theta, m)$ with respect to $w = (v, J, \vartheta, n)$ satisfies

$$\langle(\mathsf{N}' + \lambda_{\mathsf{E}}\mathsf{E}' + \lambda_G G')(z)|w\rangle = 0.$$

We have

$$\begin{aligned}\langle\mathsf{N}'(z)|w\rangle &= \int_\Omega \partial_\theta \eta\, \vartheta\, \mathrm{d}x = \int_\Omega \frac{\vartheta}{\theta}\, \mathrm{d}x,\\ \langle\lambda_{\mathsf{E}}\mathsf{E}'(z)|w\rangle &= \int_\Omega \lambda_{\mathsf{E}}(u\cdot v + F : J + \vartheta + \nabla m : \nabla n)\, \mathrm{d}x,\\ \langle\lambda_G G'(z)|w\rangle &= \int_\Omega \lambda_G\, m\cdot n\, \mathrm{d}x.\end{aligned}$$

This yields the relation

$$0 = \int_\Omega [(1/\theta + \lambda_{\mathsf{E}})\vartheta + \lambda_{\mathsf{E}}(u\cdot v + F : J - \Delta m\cdot n) + \lambda_G\, m\cdot n]\, \mathrm{d}x, \tag{2.8}$$

where we employed the boundary condition $\partial_\nu m = 0$ to derive

$$\int_\Omega \nabla m : \nabla n\, \mathrm{d}x = \int_\Omega -\Delta m\cdot n\, \mathrm{d}x.$$

Now, setting $(v, J, n) = (0, 0, 0)$ in (2.8), it follows that $\lambda_{\mathsf{E}} = -1/\theta$, as ϑ can be arbitrary. Notice that $\lambda_{\mathsf{E}} \in \mathbb{R}$, hence $\theta = \theta_*$ is constant and $\lambda_{\mathsf{E}} < 0$. Similarly, setting $(u, F) = (0, 0)$, we see that m solves the equation $-\lambda_{\mathsf{E}}\Delta m + \lambda_G m = 0$, with boundary condition $\partial_\nu m = 0$. Moreover, $|m| = 1$ on Ω. We can then conclude that

$$\lambda_G = \lambda_G|m|^2 = \lambda_{\mathsf{E}}\Delta m\cdot m = \lambda_{\mathsf{E}}\nabla\cdot[(\nabla m)m] - \lambda_{\mathsf{E}}|\nabla m|^2 = -\lambda_{\mathsf{E}}|\nabla m|^2, \tag{2.9}$$

where we used the fact that $(\nabla m)m = 0$ due to $|m| = 1$. This implies

$$\Delta m + |\nabla m|^2 m = 0,$$

hence m satisfies the harmonic map equation (2.7). Therefore, the critical points $z = (u, F, \theta, m)$ of the constrained entropy functional correspond exactly to the equilibria $\mathcal{E}$ of the system.

Meanwhile, let

$$\mathsf{H}(z) := (\mathsf{N}'' + \lambda_{\mathsf{E}}\mathsf{E}'' + \lambda_G G'')(z)$$

be the second variation of N at a generic point $z = (u, F, \theta, m)$. A direct computation yields

$$(\mathsf{H}(z)w|w)_\Omega = \int_\Omega [-\frac{1}{\theta^2}\vartheta^2 + \lambda_{\mathsf{E}}(|v|^2 + |J|^2 + |\nabla n|^2) + \lambda_G\, |n|^2]\, \mathrm{d}x,$$

where $w = (v, J, \vartheta, n)$. At a critical point $z_* = (0, 0, \theta_*, m_*)$, we obtain, in conjunction with the relation $\lambda_G = -\lambda_{\mathsf{E}}|\nabla m_*|^2$, see (2.9),

$$(\mathsf{H}(z_*)w|w)_\Omega = \int_\Omega [-\frac{1}{\theta_*^2}\vartheta^2 - \frac{1}{\theta_*}(|v|^2 + |J|^2 + |\nabla n|^2 - |\nabla m_*|^2|n|^2]\,\mathrm{d}x.$$

A moment of reflection shows that $N(\mathsf{E}'(z_*))$ and $N(G'(z_*))$, the null space of $\mathsf{E}'(z_*)$ and $G'(z_*)$, respectively, is given by

$$N(\mathsf{E}'(z_*)) = \{(v, J, \vartheta, n) : \int_\Omega \vartheta\,\mathrm{d}x = 0, \int_\Omega \nabla m_* : \nabla n\,\mathrm{d}x = 0\},$$
$$N(G'(z_*)) = \{(v, J, \vartheta, n) : \; m_* \cdot n = 0 \;\text{ in }\; \Omega\}.$$

We note that the condition $m_* \cdot n = 0$ in Ω implies $\int_\Omega \nabla m_* : \nabla n\,\mathrm{d}x = 0$. This follows from the relation

$$0 = |\nabla m_*|^2 m_* \cdot n = -\Delta m_* \cdot n$$

and an integration by parts. Hence, we have

$$N_* := N(\mathsf{E}'(z_*)) \cap N(G'(z_*)) = \{(v, F, \vartheta, n) : \int_\Omega \vartheta\,\mathrm{d}x = 0, \;\; m_* \cdot n = 0 \;\text{ in }\; \Omega\}.$$

This yields

$$\begin{aligned}(\mathsf{H}(z_*)w|w)_\Omega &= -\frac{1}{\theta_*}\int_\Omega \left(\frac{\vartheta^2}{\theta_*} + |v|^2 + |J|^2 + |\nabla n|^2 - |\nabla m_*|^2|n|^2\right)\mathrm{d}x,\\ w &= (v, J, \vartheta, n) \in N_*.\end{aligned}$$

We conclude that $\mathsf{H}(z_*)|_{N_*}$, the restriction of $\mathsf{H}(z_*)$ on N_*, is negative semi-definite iff

$$\int_\Omega (|\nabla n|^2 - |\nabla m_*|^2|n|^2)\,\mathrm{d}x \geq 0, \quad n \in C_c^\infty(\Omega; \mathbb{R}^3). \tag{2.10}$$

In case relation (2.10) holds for all n satisfying $m_* \cdot n = 0$ in Ω, m_* is called a stable harmonic map, see for instance [24, formula (1.5)]. Note that (2.10) holds if $m_* \in \mathbb{S}^2$ is constant. This shows that the validity of the relation (2.10) is necessary for the constrained entropy functional to have a (local) maximum at a critical point. We refer to [8, Theorem 26.2] for more background on extrema for constrained problems in infinite-dimensional Banach spaces.

Summarizing, we have (informally) shown the following result.

- The equilibria of (1.1) are precisely the critical points of the entropy functional with prescribed energy.
- The condition (2.10) is necessary for the constrained entropy functional to have a local maximum at a critical point. This always holds true in case $m_* \in \mathbb{S}^2$ is constant.

Remark 2.3. (a) We notice that equilibria which are (local) maxima of the constrained entropy functional are the ultimate states where the system is evolving toward. These are necessarily (locally) stable, as entropy can then no longer increase.

(b) It is stated in [16, Lemma 5.2], see also [17, Lemma 1] and [32, Lemma 12.2.4], that the nonlinear problem (2.7) admits only constant solutions $m_* \in \mathbb{S}^2$. However, this assertion is not correct in the form stated, as the following example shows: Let $\Omega = \{x \in \mathbb{R}^3 : 0 < r_1 < |x| < r_2\}$ and $m : \Omega \to \mathbb{S}^2$ be defined by $m_*(x) = x/|x|$. Then, m_* is a (non-constant) solution of (2.7).

3. The functional analytic setting

Following the formulation in [10, Section 2], we rewrite the system (1.1) as

$$\begin{cases} \partial_t u + u\cdot\nabla u - \nabla\cdot(\mu(\theta)\nabla u) + \nabla\pi = -\nabla\cdot(\nabla m \odot \nabla m) + \nabla\cdot(FF^\top) & \text{in} \quad \Omega, \\ \nabla\cdot u = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ \partial_t F + u\cdot\nabla F - \nabla\cdot(\kappa(\theta)\nabla F) = (\nabla u)^\top F & \text{in} \quad \Omega, \\ F = 0 & \text{on} \quad \partial\Omega, \\ \partial_t\theta + u\cdot\nabla\theta - \nabla\cdot(K(z)\nabla\theta) = \mu(\theta)|\nabla u|^2 + \kappa(\theta)|\nabla F|^2 & \\ \quad + \alpha(\theta)|\Delta m + |\nabla m|^2 m|^2 & \text{in} \quad \Omega, \\ \nu\cdot(K(z)\nabla\theta) = 0 & \text{on} \quad \partial\Omega, \\ \partial_t m - (\alpha(\theta)I_3 - \beta(\theta)\mathsf{M}(m))\Delta m = \alpha(\theta)|\nabla m|^2 m - u\cdot\nabla m & \text{in} \quad \Omega, \\ \partial_\nu m = 0 & \text{on} \quad \partial\Omega, \\ (u(0), F(0), m(0), \theta(0)) = (u_0, F_0, m_0, \theta_0) & \text{in} \quad \Omega, \end{cases} \tag{3.1}$$

where $z = (u, F, \theta, m)$,

$$\mathsf{M}(m) = \begin{bmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{bmatrix}, \qquad \text{with} \quad m = (m_1, m_2, m_3).$$

One readily verifies that $m \times \Delta m = \mathsf{M}(m)\Delta m$. Notice that the constraint $|m| = 1$ is dropped in (3.1). It will be shown later that the condition $|m| \equiv 1$ is in fact preserved, provided $|m_0| = 1$.

Given any $\widetilde{z} = (\widetilde{u}, \widetilde{F}, \widetilde{\theta}, \widetilde{m}) \in C^1(\overline{\Omega}; \mathbb{R}^3 \times \mathbb{R}^9 \times \mathbb{R} \times \mathbb{R}^3)$ (where we have identified $\mathbb{M}^3$ with $\mathbb{R}^9$), we introduce the operators

$$\begin{aligned} A^1(\widetilde{\theta})u &:= -P_H(\nabla\cdot(\mu(\widetilde{\theta})\nabla u)), \\ A^2(\widetilde{\theta})F &:= -\nabla\cdot(\kappa(\widetilde{\theta})\nabla F), \\ A^3(\widetilde{z})\theta &:= -\nabla\cdot(K(\widetilde{z})\nabla\theta), \\ A^4(\widetilde{\theta}, \widetilde{m})m &:= -(\alpha(\widetilde{\theta})I_3 - \beta(\widetilde{\theta})\mathsf{M}(\widetilde{m}))\Delta m - \alpha(\widetilde{\theta})|\nabla\widetilde{m}|^2 m. \end{aligned}$$

Here, $P_H : L_p(\Omega; \mathbb{R}^3) \to L_{p,\sigma}(\Omega; \mathbb{R}^3)$ denotes the Helmholtz projection.

For later use, we show that the principal part $A^4_\sharp(\widetilde{\theta}, \widetilde{m}) = (\alpha(\widetilde{\theta})I_3 - \beta(\widetilde{\theta})\mathsf{M}(\widetilde{m}))\Delta$ of $A^4(\widetilde{\theta}, \widetilde{m})$ is (uniformly) normally elliptic, see for instance [32, Definition 6.1.1].

For this, let $(\widetilde{\theta}, \widetilde{m}) \in C(\overline{\Omega}) \times C(\overline{\Omega}; \mathbb{R}^3)$ be given. The symbol $SA^4_\sharp(\widetilde{\theta}, \widetilde{m})(x, \xi)$ of $A^4_\sharp(\widetilde{\theta}, \widetilde{m})$ is given by

$$SA^4_\sharp(\widetilde{\theta}, \widetilde{m})(x, \xi) = \big(\alpha(\widetilde{\theta}(x))I_3 - \beta(\widetilde{\theta}(x))\mathsf{M}(\widetilde{m}(x))\big)|\xi|^2, \quad (x, \xi) \in \overline{\Omega} \times \mathbb{R}^3.$$

For $|\xi| = 1$, we obtain for the spectrum σ (i.e., the eigenvalues)

$$\sigma(SA^4_\sharp(\widetilde{\theta}, \widetilde{m})(x, \xi)) = \{\alpha(\widetilde{\theta}(x)),\ \alpha(\widetilde{\theta}(x)) \pm i\beta(\widetilde{\theta}(x))|\widetilde{m}(x)|\}, \quad (x, \xi) \in \overline{\Omega} \times \mathbb{S}^2.$$

Hence, for each $(\widetilde{\theta}, \widetilde{m}) \in C(\overline{\Omega}) \times C(\overline{\Omega}; \mathbb{R}^3)$ and $(x, \xi) \in \overline{\Omega} \times \mathbb{R}^3$, the spectrum of $SA^4_\sharp(\widetilde{\theta}, \widetilde{m})(x, \xi)$ is contained in a sector $\Sigma_\vartheta = \{z \in \mathbb{C}\backslash\{0\} : |\arg z| < \vartheta\}$ with opening angle ϑ, satisfying

$$\tan(\vartheta) \le \frac{\max_{x\in\overline{\Omega}} |\beta(\widetilde{\theta}(x))||\widetilde{m}(x)|}{\min_{x\in\overline{\Omega}} \alpha(\widetilde{\theta}(x))} \le \frac{\max_{x\in\overline{\Omega}} |\beta(\widetilde{\theta}(x))||\widetilde{m}(x)|}{\underline{\alpha}} \le M,$$

where $\underline{\alpha} > 0$ is the constant introduced in (2.1) and $M \ge 0$ is an appropriate constant. Hence, $\vartheta < \pi/2$.

Setting

$$[\mathcal{C}^1(\widetilde{m})m]_i = \partial_i\widetilde{m} \cdot \Delta m + \nabla\widetilde{m} : \partial_i \nabla m, \quad i = 1, 2, 3,$$

it follows that

$$\begin{aligned} &\mathcal{C}^1(m)m = \nabla \cdot (\nabla m \odot \nabla m), && m \in W^2_p(\Omega; \mathbb{R}^3), \\ &\mathcal{C}^1(\widetilde{m}) \in \mathcal{L}(W^2_p(\Omega; \mathbb{R}^3), L_p(\Omega; \mathbb{R}^3)), && \widetilde{m} \in C^1(\overline{\Omega}; \mathbb{R}^3). \end{aligned}$$

For $(\widetilde{\theta}, \widetilde{m}) \in C(\overline{\Omega}) \times C^2(\overline{\Omega}; \mathbb{R}^3)$, we set

$$\mathcal{C}^3(\widetilde{\theta}, \widetilde{m})m := -\alpha(\widetilde{\theta})(\Delta\widetilde{m} + |\nabla\widetilde{m}|^2\widetilde{m}) \cdot (\Delta m + |\nabla\widetilde{m}|^2 m).$$

One readily verifies that

$$\mathcal{C}^3(\widetilde{\theta}, \widetilde{m}) \in \mathcal{L}(C^2(\overline{\Omega}; \mathbb{R}^3), C(\overline{\Omega}, \mathbb{R}^3)).$$

We now introduce a functional analytic setting to study problem (3.1). For this, let

$$X_0 := L_{p,\sigma}(\Omega; \mathbb{R}^3) \times L_p(\Omega; \mathbb{M}^3) \times L_p(\Omega; \mathbb{R}) \times W^1_p(\Omega; \mathbb{R}^3), \quad 1 < p < \infty.$$

Here, $L_{p,\sigma}(\Omega; \mathbb{R}^3) := P_H(L_p(\Omega; \mathbb{R}^3))$ is the space of all solenoidal vector fields in $L_p(\Omega; \mathbb{R}^3)$ with $P_H : L_p(\Omega; \mathbb{R}^3) \to L_{p,\sigma}(\Omega; \mathbb{R}^3)$ the Helmholtz projection. For all $s \ge 0$, we define

$$W^s_{p,\sigma}(\Omega; \mathbb{R}^3) : = W^s_p(\Omega; \mathbb{R}^3) \cap L_{p,\sigma}(\Omega; \mathbb{R}^3).$$

Moreover, we set

$$\begin{aligned} X_1^1 &:= \{u \in W^2_{p,\sigma}(\Omega;\mathbb{R}^3) : u = 0 \text{ on } \partial\Omega\}, \\ X_1^2 &:= \{F \in W^2_p(\Omega;\mathbb{M}^3) : F = 0 \text{ on } \partial\Omega\}, \\ X_1^3 &:= \{\theta \in W^2_p(\Omega)\}, \\ X_1^4 &:= \{m \in W^3_p(\Omega;\mathbb{R}^3) : \partial_\nu m = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

For $X_1 := X_1^1 \times X_1^2 \times X_1^3 \times X_1^4$, we introduce the space of initial data as

$$X_{\gamma,\mu} : = (X_0, X_1)_{\mu-1/p,p}$$

for $\mu \in (1/p, 1]$. In the following, we assume

$$p > 5 \quad \text{and} \quad \mu > \frac{1}{2} + \frac{5}{2p}, \tag{3.2}$$

which ensures that the embedding

$$W_p^{j+2\mu-2/p}(\Omega) \hookrightarrow C^{j+1}(\overline{\Omega}), \qquad j = 0, 1, \tag{3.3}$$

holds true. Observe that by [3, Theorem 3.4] and [37, Theorem 4.3.3],

$$(u, F, \theta, m) \in X_{\gamma,\mu} \;\Leftrightarrow\; \begin{cases} u \in W^{2\mu-2/p}_{p,\sigma}(\Omega;\mathbb{R}^3) & \text{and} \quad u = 0 \quad \text{on} \quad \partial\Omega, \\ F \in W^{2\mu-2/p}_{p,\sigma}(\Omega;\mathbb{M}^3) & \text{and} \quad F = 0 \quad \text{on} \quad \partial\Omega, \\ \theta \in W^{2\mu-2/p}_{p}(\Omega), & \\ m \in W^{1+2\mu-2/p}_{p}(\Omega;\mathbb{R}^3) & \text{and} \quad \partial_\nu m = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Given any $\widetilde{z} = (\widetilde{u}, \widetilde{F}, \widetilde{\theta}, \widetilde{m}) \in X_{\gamma,\mu}$, the operator

$$\mathsf{A}(\widetilde{z}) := \begin{bmatrix} A^1(\widetilde{\theta}) & 0 & 0 & P_H \mathcal{C}^1(\widetilde{m}) \\ 0 & A^2(\widetilde{\theta}) & 0 & 0 \\ 0 & 0 & A^3(\widetilde{z}) & \mathcal{C}^3(\widetilde{\theta},\widetilde{m}) \\ 0 & 0 & 0 & A^4(\widetilde{\theta},\widetilde{m}) \end{bmatrix} \tag{3.4}$$

satisfies $\mathsf{A}(\widetilde{z}) \in \mathcal{L}(X_1, X_0)$.

In addition, given $z = (u, F, \theta, m)$, we introduce the boundary operator $\mathsf{B}(\widetilde{z})$, defined by

$$\mathsf{B}(\widetilde{z})z = \nu \cdot \mathrm{tr}_{\partial\Omega}\left((K(\widetilde{z})\nabla\theta)\right), \tag{3.5}$$

where $\mathrm{tr}_{\partial\Omega}$ is the boundary trace operator.

Finally, we set

$$\mathsf{F}(z) = \begin{bmatrix} P_H\left[\nabla\cdot(FF^{\mathsf{T}}) - (u\cdot\nabla u)\right] \\ (\nabla u)^{\mathsf{T}}F - u\cdot\nabla F \\ \upmu(\theta)|\nabla u|^2 + \kappa(\theta)|\nabla F|^2 - u\cdot\nabla\theta \\ -u\cdot\nabla m \end{bmatrix}. \tag{3.6}$$

Using the notation introduced in (3.4), (3.5), and (3.6), we can restate the nonlinear system (3.1) in the condensed form

$$\begin{cases} \partial_t z + \mathsf{A}(z)z = \mathsf{F}(z) & \text{in} \quad \Omega, \\ \mathsf{B}(z)z = 0 & \text{on} \quad \partial\Omega, \\ z(0) = z_0 & \text{in} \quad \Omega. \end{cases} \tag{3.7}$$

For notational brevity, given any $T \in (0, \infty]$, we define

$$\begin{aligned} &\mathbb{E}_{0,\mu}(J_T) := L_{p,\mu}(J_T; X_0), \qquad \mathbb{E}_{1,\mu}(J_T) := W^1_{p,\mu}(J_T; X_0) \cap L_p(J_T; X_1), \\ &\mathbb{B}_\mu(J_T) := BUC(J_T; X_{\gamma,\mu}), \\ &\mathbb{F}_\mu(J_T) := W^{1/2-1/2p}_{p,\mu}(J_T; L_p(\partial\Omega)) \cap L_{p,\mu}(J_T; W^{1-1/p}_p(\partial\Omega)), \end{aligned}$$

and

$$Y_{\gamma,\mu} = W^{2\mu-1-3/p}_p(\partial\Omega).$$

For future analysis, we also introduce the spaces with vanishing trace at $t = 0$:

$$\begin{aligned} {}_0\mathbb{E}_{1,\mu}(J_T) &:= {}_0\mathbb{E}^1_{1,\mu}(J_T) \times {}_0\mathbb{E}^2_{1,\mu}(J_T) \times {}_0\mathbb{E}^3_{1,\mu}(J_T) \times {}_0\mathbb{E}^4_{1,\mu}(J_T) \\ &:= \{(z_1, z_2, z_3, z_4) \in \mathbb{E}_1(J_T) : \gamma_0(z_1, z_2, z_3, z_4) = (0, 0, 0, 0)\}, \\ {}_0\mathbb{B}_\mu(J_T) &:= \{z \in \mathbb{B}_\mu(J_T) : \gamma_0 z = 0\}, \\ {}_0\mathbb{F}_\mu(J_T) &:= \{g \in \mathbb{F}_\mu(J_T) : \gamma_0 g = 0\}, \end{aligned}$$

where γ_0 denotes the trace operator at $t = 0$.

Lemma 3.1. *Suppose that (p, μ) satisfy the Assumption (3.2). Let $T \in (0, \infty]$. Then, we have*

(a) $\mathbb{E}_{1,\mu}(J_T) \hookrightarrow \mathbb{B}_\mu(J_T)$. *The embedding constant for the embedding ${}_0\mathbb{E}_{1,\mu}(J_T) \hookrightarrow {}_0\mathbb{B}_\mu(J_T)$ is independent of T.*
(b) *The trace operator $\gamma_0 \in \mathcal{L}(\mathbb{E}_{1,\mu}(J_T), X_{\gamma,\mu})$ has a bounded right inverse $\gamma_0^c \in \mathcal{L}(X_{\gamma,\mu}, \mathbb{E}_{1,\mu}(J_T))$.*
(c) *For each $z_0 \in X_{\gamma,\mu}$, there exists a function $z_* \in \mathbb{E}_{1,\mu}(\mathbb{R}_+)$ such that $z_*(0) = z_0$.*
(d) $\mathbb{F}_\mu(J_T) \hookrightarrow BUC(J; Y_{\gamma,\mu}) \hookrightarrow BUC(J \times \partial\Omega)$. *Thus, $\mathbb{F}_\mu(J_T)$ is a multiplication algebra.*

Proof. Although the properties listed above are known, for the reader's convenience, we include a proof nonetheless.

(a) By [28, Lemma 2.5], there exists an extension operator $\mathcal{E}_{J_T} \in \mathcal{L}(\mathbb{E}_{1,\mu}(J_T), \mathbb{E}_{1,\mu}(\mathbb{R}_+))$. Moreover, one shows that $\mathbb{E}_{1,\mu}(\mathbb{R}_+)$ is continuously translation invariant. The first assertion follows now from [4, Proposition III.1.4.2].

By Proposition A.4, there exists an extension operator $\mathcal{E}^0_{J_T} \in \mathcal{L}({}_0\mathbb{E}_{1,\mu}(J_T), {}_0\mathbb{E}_{1,\mu}(\mathbb{R}_+))$ whose norm is independent of T. The second assertion follows then from the commutativity of the diagram

$$\begin{array}{ccc} {}_0\mathbb{E}_{1,\mu}(J_T) & \stackrel{\mathcal{E}^0_{J_T}}{\longrightarrow} & {}_0\mathbb{E}_{1,\mu}(\mathbb{R}_+) \\ \downarrow & & \downarrow \\ {}_0\mathbb{B}_{\mu}(J_T) & \stackrel{\mathcal{R}_J}{\longleftarrow} & {}_0\mathbb{B}_{\mu}(\mathbb{R}_+), \end{array}$$

where $\mathcal{R}_J$ denotes the restriction operator.

(b) Pick any $\widetilde{z} = (\widetilde{u}, \widetilde{F}, \widetilde{\theta}, \widetilde{m}) \in X_{\gamma,\mu}$. Then, we define for $z_0 \in (u_0, F_0, \theta_0, m_0) \in X_{\gamma,\mu}$

$$(\gamma_0^c z_0)(t) := \left(e^{-t(I-A^1(\widetilde{\theta}))}u_0, e^{-t(I-A^2(\widetilde{\theta}))}F_0, \mathcal{R}_\Omega e^{-t(I-\Delta)}\mathcal{E}_\Omega\theta_0, e^{-t(\omega I-A^4(\widetilde{\theta},\widetilde{m}))}m_0\right), \quad t \in J,$$

where $D(A^i(\widetilde{z})) := X_1^i$ for $i \in \{1, 2, 4\}$, with $\widetilde{z}$ interpreted appropriately, and ω is a sufficiently large positive constant, to be specified below.

Moreover, $\mathcal{E}_\Omega \in \mathcal{L}(W_p^{1+2\mu-2/p}(\Omega), W_p^{1+2\mu-2/p}(\mathbb{R}^3))$ denotes an appropriate extension operator, $\mathcal{R}_\Omega$ is the restriction operator, and Δ is the Laplacian defined on $\mathbb{R}^3$.

It follows from the maximal regularity result in Proposition 4.1 and [32, Proposition 3.5.2(ii)] that, for each $\widetilde{z} \in X_{\gamma,\mu}$, the operators $\omega I - A^i(\widetilde{z})$, $i \in \{1, 2, 4\}$, generate a strongly continuous exponentially stable analytic semigroup on X_0^i, where $\omega = 1$ for $i \in \{1, 2\}$, and ω is sufficiently large for $i = 4$. The assertion follows then from [37, Theorem 1.14.5].

(c) This follows by choosing $T = \infty$ in (b) and setting $z_* = \gamma_0^c z_0$.

(d) The first embedding follows from [28, formula (4.10)], and the second one from (3.2) and Sobolev embedding. □

4. Linearized problems

Before investigating the nonlinear system (3.7), we first study some related linear problems. We start with the system

$$\begin{cases} \partial_t z + \mathsf{A}(\widetilde{z})z = \mathsf{f}(t) & \text{in} \quad \Omega, \\ \mathsf{B}(\widetilde{z})z = \mathsf{g}(t) & \text{on} \quad \partial\Omega, \\ z(0) = z_0 & \text{in} \quad \Omega, \end{cases} \tag{4.1}$$

where $\widetilde{z} \in X_{\gamma,\mu}$.

Proposition 4.1. *Assume (2.1) and (3.2). Let $\widetilde{z} \in X_{\gamma,\mu}$ and $T > 0$ be given.*

(a) *For every $(\mathsf{f}, \mathsf{g}, z_0) \in \mathbb{E}_{0,\mu}(J_T) \times \mathbb{F}_\mu(J_T) \times X_{\gamma,\mu}$, where g satisfies the compatibility condition $\mathsf{B}(\widetilde{z})z_0 = \gamma_0\mathsf{g}$, the linear initial boundary value problem (4.1) admits a unique solution $z \in \mathbb{E}_{1,\mu}(J_T)$. Moreover,*

$$\mathsf{L}(\widetilde{z}) := (\mathsf{A}(\widetilde{z}), \mathsf{B}(\widetilde{z}), \gamma_0) \in \mathcal{L}is(\mathbb{E}_{1,\mu}(J_T), \mathbb{D}_\mu(\widetilde{z}, T)),$$

where $\mathbb{D}_\mu(\widetilde{z}, T) := \{(\mathsf{f}, \mathsf{g}, z_0) \in \mathbb{E}_{0,\mu}(J_T) \times \mathbb{F}_\mu(J_T) \times X_{\gamma,\mu} : \mathsf{B}(\widetilde{z})z_0 = \gamma_0 \mathsf{g}\}$.

(b) *For each* $\widetilde{z} \in X_{\gamma,\mu}$ *and* $(\mathsf{f}, \mathsf{g}) \in \mathbb{E}_{0,\mu}(J_T) \times {}_0\mathbb{F}_\mu(J_T)$*, let* $z =: \mathsf{S}(\widetilde{z})(\mathsf{f}, \mathsf{g})$ *be the (unique) solution of* (4.1) *with* $z_0 = 0$*. Then,*

$$[\widetilde{z} \mapsto \mathsf{S}(\widetilde{z})] \in C^1(X_{\gamma,\mu}, \mathcal{L}(\mathbb{E}_{0,\mu}(J_T) \times {}_0\mathbb{F}_\mu(J_T), {}_0\mathbb{E}_{1,\mu}(J_T)). \tag{4.2}$$

Moreover, given any $T_* > 0$*, the norm of* S *is uniform in* $T \in (0, T_*]$.

Proof. (a) The proof is based on the upper triangular structure of A and the results in [32, Theorems 6.3.2, 6.3.3 and 7.3.2]. Let

$$\mathsf{f} = (f_1, f_2, f_3, f_4) \in \mathbb{E}_{0,\mu}(J_T), \quad \mathsf{g} \in \mathbb{F}_\mu(J_T), \quad z_0 \in X_{\gamma,\mu}$$

be given, where g satisfies the compatibility condition $\mathsf{B}(\widetilde{z})z_0 = \gamma_0 \mathsf{g}$. We first solve the equation for m:

$$\begin{aligned} \partial_t m + A^4(\widetilde{\theta}, \widetilde{m})m &= f_4 && \text{in} \quad \Omega, \\ \partial_\nu m &= 0 && \text{on} \quad \partial\Omega, \\ m(0) &= m_0 && \text{in} \quad \Omega, \end{aligned}$$

and obtain a unique solution

$$m \in L_{p,\mu}(J_T; W_p^3(\Omega; \mathbb{R}^3)) \cap W_{p,\mu}^1(J_T; W_p^1(\Omega; \mathbb{R}^3))$$

by means of [32, Theorem 6.3.3]. In view of (3.3),

$$(\mathcal{C}^1(\widetilde{m})m, 0, \mathcal{C}^3(\widetilde{\theta}, \widetilde{m})m, 0) \in \mathbb{E}_{0,\mu}(J_T).$$

Therefore, the remaining equations of (4.1) can be rewritten as

$$\begin{aligned} \partial_t u + A^1(\widetilde{\theta})u &= f_1 - \mathcal{C}^1(\widetilde{m})m && \text{in} \quad \Omega, \\ u &= 0 && \text{on} \quad \partial\Omega, \\ \partial_t F + A^2(\widetilde{\theta})F &= f_2 && \text{in} \quad \Omega, \\ F &= 0 && \text{on} \quad \partial\Omega, \\ \partial_t \theta + A^3(\widetilde{z})\theta &= f_3 - \mathcal{C}^3(\widetilde{\theta}, \widetilde{m})m && \text{in} \quad \Omega, \\ \nu \cdot \mathrm{tr}_{\partial\Omega}(K(\widetilde{z})\nabla\theta) &= \mathsf{g} && \text{on} \quad \partial\Omega, \\ (u(0), F(0), \theta(0)) &= (u_0, F_0, \theta_0) && \text{in} \quad \Omega. \end{aligned}$$

The existence and uniqueness of a solution then follows from [32, Theorems 6.3.2 and 7.3.2].

(b) Lemma B.1 implies that

$$[\widetilde{z} \mapsto (\mathsf{A}(\widetilde{z}), \mathsf{B}(\widetilde{z}))] \in C^1(X_{\gamma,\mu}, \mathcal{L}({}_0\mathbb{E}_{1,\mu}(J_T), \mathbb{E}_{0,\mu}(J_T) \times {}_0\mathbb{F}_\mu(J_T))).$$

The continuous differentiability of the map S follows from the following diagram:

$$\begin{array}{ccccc} \widetilde{z} & \mapsto & (\partial_t + \mathsf{A}(\widetilde{z}), \mathsf{B}(\widetilde{z})) & \mapsto & (\partial_t + \mathsf{A}(\widetilde{z}), \mathsf{B}(\widetilde{z}))^{-1} \\ X_{\gamma,\mu} & \to & \mathcal{L}({}_0\mathbb{E}_{1,\mu}(J_T), \mathbb{E}_{0,\mu}(J_T) \times {}_0\mathbb{F}_\mu(J_T)) & \to & \mathcal{L}(\mathbb{E}_{0,\mu}(J_T) \times {}_0\mathbb{F}_\mu(J_T), {}_0\mathbb{E}_{1,\mu}(J_T)). \end{array}$$

Fix $T_* > 0$. For any $0 < T \le T_*$, it follows from Proposition A.4 that there exists an extension map $\mathcal{E}_{J_T} : \mathbb{X}(J_T) \to \mathbb{X}(J_{T_*})$ with $\mathbb{X} \in \{\mathbb{E}_{0,\mu}, {}_0\mathbb{E}_{1,\mu}, {}_0\mathbb{F}_\mu\}$. Moreover, the norm of $\mathcal{E}_{J_T}$ is uniform in $T \in (0, T_*]$. Given any $(\mathsf{f}, \mathsf{g}) \in \mathbb{E}_{0,\mu}(J_T) \times {}_0\mathbb{F}_\mu(J_T)$, let $(\hat{\mathsf{f}}, \hat{\mathsf{g}}) = (\mathcal{E}_{J_T}\mathsf{f}, \mathcal{E}_{J_T}\mathsf{g})$. Part (a) implies that one can find a unique function $\hat{z} \in {}_0\mathbb{E}_{1,\mu}(J_{T_*})$ such that $\hat{z} = \mathsf{S}(\hat{\mathsf{f}}, \hat{\mathsf{g}})$. Put $z = \hat{z}|_{[0,T]}$. It is clear that $\mathsf{L}(\widetilde{z})z = (\mathsf{f}, \mathsf{g}, 0)$. Direct computations yield

$$\begin{aligned} \|z\|_{\mathbb{E}_{1,\mu}(J_T)} \le \|\hat{z}\|_{\mathbb{E}_{1,\mu}(J_{T_*})} &\le C\left(\|\hat{\mathsf{f}}\|_{\mathbb{E}_{0,\mu}(J_{T_*})} + \|\hat{\mathsf{g}}\|_{{}_0\mathbb{F}_\mu(J_{T_*})}\right) \\ &\le C\left(\|\mathsf{f}\|_{\mathbb{E}_{0,\mu}(J_T)} + \|\mathsf{g}\|_{{}_0\mathbb{F}_\mu(J_T)}\right) \end{aligned}$$

with a constant C that is independent of $T \in (0, T_*]$. Therefore, the norm of $\mathsf{S}(\widetilde{z})$ is uniform in $T \in (0, T_*]$. □

Next, we consider the non-autonomous linear system

$$\begin{cases} \partial_t z + \mathsf{A}(z_*(t))z = \mathsf{f}(t) & \text{in} \quad \Omega, \\ \mathsf{B}(z_*(t))z = \mathsf{g}(t) & \text{on} \quad \partial\Omega, \\ z(0) = z_0 & \text{in} \quad \Omega, \end{cases} \tag{4.3}$$

where $z_* \in \mathbb{E}_{1,\mu}(J_T)$ is given.

Proposition 4.2. *Assume (2.1) and (3.2). Let $T > 0$ and $z_* \in \mathbb{E}_{1,\mu}(J_T)$ be given. Then, the system (4.3) has a unique solution $z = S(\mathsf{f}, \mathsf{g}, z_0) \in \mathbb{E}_{1,\mu}(J_T)$ if and only if*

$$(\mathsf{f}, \mathsf{g}, z_0) \in \mathbb{D}_\mu(z_*, T) := \{\mathbb{E}_{0,\mu}(J_T) \times \mathbb{F}_\mu(J_T) \times X_{\gamma,\mu} : \mathsf{B}(z_*(0))z_0 = \mathsf{g}(0)\}.$$

In this case, there is a constant $c_1 = c_1(T) > 0$ such that

$$\|z\|_{\mathbb{E}_{1,\mu}(J_T)} \le c_1(T)\big(\|\mathsf{f}\|_{\mathbb{E}_{0,\mu}(J_T)} + \|\mathsf{g}\|_{\mathbb{F}_\mu(J_T)} + \|z_0\|_{X_{\gamma,\mu}}\big). \tag{4.4}$$

Given any $T_* > 0$, the constant $c_1(T)$ is uniform in $T \in (0, T_*]$ in case $\mathsf{g} \in {}_0\mathbb{F}_\mu(J_T)$ and $z_0 = 0$.

Proof. Let $z_* \in \mathbb{E}_{1,\mu}(J_T)$ be given. In the following, we use the notation

$$\mathsf{A}_*(t) := \mathsf{A}(z_*(t)), \quad \mathsf{B}_*(t) := \mathsf{B}(z_*(t)), \quad \mathsf{S}_*(t) := \mathsf{S}(z_*(t)), \quad K_*(t) := K(z_*(t))$$

for $t \in [0, T]$, where the solution operator S is defined in Proposition 4.1. By Lemma 3.1(a), we know that $z_* \in C([0, T]; X_{\gamma,\mu})$ and therefore, the set $\{z_*(s) : s \in [0, T]\} \subset X_{\gamma,\mu}$ is compact. We conclude from (4.2) that there exists a constant $M = M(T) > 0$ such that for $\nu \in \{\mu, 1\}$

$$\|\mathsf{S}_*(s)\|_{\mathcal{L}(\mathbb{E}_{0,\nu}(J_T) \times {}_0\mathbb{F}_\nu(J_T), {}_0\mathbb{E}_{1,\nu}(J_T))} \le M, \quad s \in [0, T]. \tag{4.5}$$

By Lemma B.1 and absolute continuity of the integral $\|\mathrm{tr}_{\partial\Omega} K\|_{\mathbb{F}_\mu(J_T)}$, we can find a partition

$$0 = t_0 < t_1 \cdots < t_n = T \quad \text{of} \quad [0, T]$$

such that

$$\begin{aligned} \max_{t \in I_j} \|z_*(t) - z_*(t_j)\|_{X_{\gamma,\mu}} &\le \eta, \\ \|\mathrm{tr}_{\partial\Omega} \left(K_*(\cdot) - K_*(t_j)\right) \|_{\mathbb{F}_\nu(I_j)} &\le \eta \end{aligned} \tag{4.6}$$

for a predetermined (fixed) number η, where $I_j = [t_j, t_{j+1}]$ for $j = 0, \dots, n-1$, and where we set

$$\nu := \begin{cases} \mu & \text{if} \quad j = 0, \\ 1 & \text{if} \quad j = 1, \dots, n-1. \end{cases}$$

Now it follows from (4.6), and Lemma A.5(i) that

$$\begin{aligned} \|\mathsf{A}_*(\cdot) - \mathsf{A}_*(t_j)\|_{\mathcal{L}({}_0\mathbb{E}_{1,\nu}(I_j), \mathbb{E}_{0,\nu}(I_j))} &\le 1/(4M), \\ \|\mathsf{B}_*(\cdot) - \mathsf{B}_*(t_j)\|_{\mathcal{L}({}_0\mathbb{E}_{1,\nu}(I_j), {}_0\mathbb{F}_\nu(I_j))} &\le 1/(4M) \end{aligned} \tag{4.7}$$

for $j = 0, \dots, n-1$. In fact, we will first choose a partition point t_1 so that the properties of (4.7) hold true for $I_0 = [0, t_1]$, and then partition the remaining interval $[t_1, T]$ if needed.

We will consider problem (4.3) on subintervals I_j. In the first step, we deal with the interval $I_0 = [t_0, t_1] = [0, t_1]$. In order to resolve the compatibility condition $\mathsf{B}_*(0) z_0 = \mathsf{g}(0)$, we consider the linear problem

$$\begin{cases} \partial_t w + \mathsf{A}_*(0) w = \mathsf{f}(t) & \text{in} \quad [0, t_1] \times \Omega, \\ \mathsf{B}_*(0) w = \mathsf{g}(t) & \text{on} \quad [0, t_1] \times \partial\Omega, \\ w(0) = z_0 & \text{in} \quad \Omega. \end{cases} \tag{4.8}$$

Let $w \in \mathbb{E}_{1,\mu}([0, t_1])$ be the unique solution of (4.8) (whose existence is guaranteed by Proposition 4.1) and consider the system

$$\begin{cases} \partial_t \hat{z} + \mathsf{A}_*(t) \hat{z} = \hat{\mathsf{f}}(t) & \text{in} \quad [0, t_1] \times \Omega, \\ \mathsf{B}_*(t) \hat{z} = \hat{\mathsf{g}}(t) & \text{on} \quad [0, t_1] \times \partial\Omega, \\ \hat{z}(0) = 0 & \text{in} \quad \Omega, \end{cases} \tag{4.9}$$

where

$$\begin{aligned} \hat{\mathsf{f}}(t) &= -[\mathsf{A}_*(t) - \mathsf{A}_*(0)] w(t), \\ \hat{\mathsf{g}}(t) &= -[\mathsf{B}_*(t) - \mathsf{B}_*(0)] w(t). \end{aligned}$$

Suppose $\hat{z} \in \mathbb{E}_{1,\mu}([0, t_1])$ is a solution of (4.9). Then, one verifies that the function $z_1 = w + \hat{z} \in \mathbb{E}_{1,\mu}([0, t_1])$ is a solution of (4.3) on the interval $[0, t_1]$.

Hence, it remains to show that (4.9) has a (unique) solution. For this, we first note that the necessary compatibility condition $\mathsf{B}_*(0)\hat{z}(0) = \hat{\mathsf{g}}(0)$ is satisfied. To show the solvability, we rewrite (4.9) as

$$\begin{cases} \partial_t \hat{z} + \mathsf{A}_*(0)\hat{z} + \mathsf{R}_1(t)\hat{z} = \hat{\mathsf{f}}(t) & \text{in} \quad [0, t_1] \times \Omega, \\ \mathsf{B}_*(0)\hat{z} + \mathsf{R}_2(t)\hat{z} = \hat{\mathsf{g}}(t) & \text{on} \quad [0, t_1] \times \partial\Omega, \\ \hat{z}(0) = 0 & \text{in} \quad \Omega, \end{cases} \tag{4.10}$$

where

$$\mathsf{R}_1(t) = [\mathsf{A}_*(t) - \mathsf{A}_*(0)], \quad \mathsf{R}_2(t) = [\mathsf{B}_*(t) - \mathsf{B}_*(0)], \quad t \in [0, t_1].$$

It follows from (4.5) and (4.7) that

$$\|(\mathsf{R}_1(\cdot), \mathsf{R}_2(\cdot))\mathsf{S}_*(0)\|_{\mathcal{L}(\mathbb{E}_{0,\mu}([0,t_1])\times {}_0\mathbb{F}_\mu([0,t_1]))} \le 1/2,$$

so that $\big[I + (\mathsf{R}_1(\cdot), \mathsf{R}_2(\cdot))\mathsf{S}_*(0)\big] \in \mathcal{L}(\mathbb{E}_{0,\mu}([0, t_1]) \times {}_0\mathbb{F}_\mu([0, t_1]))$ is invertible. Hence,

$$\hat{z} = \mathsf{S}_*(0)\big[I + (\mathsf{R}_1(\cdot), \mathsf{R}_2(\cdot))\mathsf{S}_*(0)\big]^{-1}(\hat{\mathsf{f}}, \hat{\mathsf{g}}) \in {}_0\mathbb{E}_{1,\mu}([0, t_1])$$

is the (unique) solution of (4.10) on the interval $[0, t_1]$. It follows that $z_1 := \hat{z} + w \in \mathbb{E}_{1,\mu}([0, t_1])$ is a solution of (4.3) on the time interval $[0, t_1]$.

Assume that there exists another solution $\widetilde{z} \in \mathbb{E}_{1,\mu}([0, t_1])$ to (4.3) on $[0, t_1]$. Then, $h = z - \widetilde{z}$ solves (4.10) with $\hat{\mathsf{f}} = 0$ and $\hat{\mathsf{g}} = 0$. The unique solvability of (4.10) implies that $h = 0$. This proves the uniqueness of a solution on $[0, t_1]$.

We can now repeat the steps above for the interval $[t_1, t_2]$. In this case, we consider the problem

$$\begin{cases} \partial_t z + \mathsf{A}_*(t_1 + t)z = \mathsf{f}(t_1 + t) & \text{in} \quad [0, t_2 - t_1] \times \Omega, \\ \mathsf{B}_*(t_1 + t)z = \mathsf{g}(t_1 + t) & \text{on} \quad [0, t_2 - t_1] \times \partial\Omega, \\ z(0) = z_1(t_1) & \text{in} \quad \Omega, \end{cases} \tag{4.11}$$

where z_1 is the function obtained in step 1. As z_1 solves (4.3) on the time interval $[0, t_1]$, the compatibility condition $\mathsf{B}_*(t_1)z(0) = \mathsf{B}_*(t_1)z_1(t_1) = \mathsf{g}(t_1)$ is satisfied. Repeating the arguments of step 1, we obtain a unique solution $z_2 \in \mathbb{E}_{1,1}([0, t_2 - t_1])$ of (4.11), since $z_1(t_1) \in X_{\gamma,1}$. Let

$$z(t) := \begin{cases} z_1(t), & 0 \le t \le t_1 \\ z_2(t - t_1), & t_1 \le t \le t_2. \end{cases}$$

As $z_1(t_1) = z_2(0)$ we conclude that $z \in W^1_1((0, t_2); X_0)$. It is then easy to see that $z \in \mathbb{E}_{1,\mu}([0, t_2])$.

We can now repeat the steps above to find a solution $z \in \mathbb{E}_{1,\mu}(J_T)$ of (4.3) on $[0, T]$. To show uniqueness, let

$$t_* := \sup\{t \in [0, T] : \text{(4.3) has a unique solution on } [0, t]\}.$$

By step 1, the set under consideration is nonempty and, therefore, t_* is well-defined. Suppose $t_* < T$. We can then repeat step 2 from above to get a unique solution on $[t_*, t_* + \delta]$ for some δ. This leads to a contradiction, as the assumption $t_* < T$ would imply that (at least) two different solutions with initial value $z(t_*)$ exist.

Given $T_* > 0$, when $z_0 = 0$ and $\mathsf{g} \in {}_0\mathbb{F}_\mu(J_T)$, the uniformity of the constant C in (4.4) with respect to $T \in (0, T_*]$ can be shown in an analogous way to Proposition 4.1(b). □

For simplicity, we will introduce the following notation

$$\mathcal{A}(z) := \mathsf{A}(z)z \quad \text{and} \quad \mathcal{B}(z) := \mathsf{B}(z)z. \tag{4.12}$$

It follows from Proposition B.3 that

$$\begin{aligned} &\mathcal{A} \in C^1(\mathbb{E}_{1,\mu}(J_T), \mathbb{E}_{0,\mu}(J_T)), \qquad && \mathcal{A}'(z_*)z = \mathsf{A}(z_*)z + [\mathsf{A}'(z_*)z]z_*, \\ &\mathcal{B} \in C^1(\mathbb{E}_{1,\mu}(J_T), \mathbb{F}_\mu(J_T)), && \mathcal{B}'(z_*)z = \mathsf{B}(z_*)z + [\mathsf{B}'(z_*)z]z_*. \end{aligned}$$

Next, we will study solvability of the linearized system

$$\begin{cases} \partial_t z + \mathcal{A}'(z_*(t))z - \mathsf{F}'(z_*(t))z = \mathsf{f}(t) & \text{in} \quad \Omega, \\ \mathcal{B}'(z_*(t))z = \mathsf{g}(t) & \text{on} \quad \partial\Omega, \\ z(0) = z_0 & \text{in} \quad \Omega, \end{cases} \tag{4.13}$$

where $z_* \in \mathbb{E}_{1,\mu}(J_T)$ and F is defined in (3.6). We obtain the following result.

Proposition 4.3. *Let $z_* \in \mathbb{E}_{1,\mu}(J_T)$ be given. Then, the linearized system* (4.13) *has a unique solution $z = \mathcal{S}(\mathsf{f}, \mathsf{g}, z_0) \in \mathbb{E}_{1,\mu}(J_T)$ if and only if*

$$(\mathsf{f}, \mathsf{g}, z_0) \in \widetilde{\mathbb{D}}_\mu(z_*, T) := \{\mathbb{E}_{0,\mu}(J_T) \times \mathbb{F}_\mu(J_T) \times X_{\gamma,\mu} : \mathcal{B}'(z_*(0))z_0 = \mathsf{g}(0)\}.$$

In this case, there is a constant $c_2 = c_2(T) > 0$ such that

$$\|z\|_{\mathbb{E}_{1,\mu}(J_T)} \le c_2(T)\big(\|\mathsf{f}\|_{\mathbb{E}_{0,\mu}(J_T)} + \|\mathsf{g}\|_{\mathbb{F}_\mu(J_T)} + \|z_0\|_{X_{\gamma,\mu}}\big).$$

Given any $T_* > 0$, the constant c_2 is independent of $T \in (0, T_*]$ in case $\mathsf{g} \in {}_0\mathbb{F}_\mu(J_T)$ and $z_0 = 0$.

Proof. We observe that for $z \in {}_0\mathbb{E}_{1,\nu}(I)$ and any interval I contained in $[0, T]$,

$$\begin{aligned} \left(\int_I \|t^{1-\nu}[\mathsf{A}'(z_*(t))z(t)]z_*(t)\|_{X_0}^p \,\mathrm{d}t\right)^{1/p} &= \left(\int_I \|[\mathsf{A}'(z_*(t))z(t)]t^{1-\nu}z_*(t)\|_{X_0}^p \,\mathrm{d}t\right)^{1/p} \\ &\le M\left(\int_I \|t^{1-\nu}z_*(t)\|_{X_1}^p \,\mathrm{d}t\right)^{1/p} \sup_{t\in I}\|z(t)\|_{X_{\gamma,\mu}} \\ &\le CM\left(\int_I \|t^{1-\nu}z_*(t)\|_{X_1}^p \,\mathrm{d}t\right)^{1/p} \|z\|_{{}_0\mathbb{E}_{1,\mu}(I)}, \end{aligned}$$

where the constants C and M do not depend on the length of I, due to Lemma 3.1(a). Here, $\nu = \mu$ in case the interval I contains 0, and $\nu = 1$ otherwise. By absolute continuity of the integral $\int_{[0,T]} \|t^{1-\nu} z_*(t)\|_{X_1}^p \, \mathrm{d}t$, we can, again, choose a partition $\{I_j : 0 \le j \le n-1\}$ of $[0, T]$ such that

$$\left(\int_{I_j} \|t^{1-\nu} z_*(t)\|_{X_1}^p \, \mathrm{d}t \right)^{1/p} \le \eta,$$

where $\eta > 0$ is a given, predetermined (small) number.

Let $z_* = (u_*, F_*, \theta_*, m_*) \in \mathbb{E}_{1,\mu}(J_T)$ be given. Then $[\mathsf{B}'(z_*)z]z_* = \mathrm{tr}_{\partial\Omega}[K'(z_*)z] \nabla\theta_*$. It follows from Lemma A.5(ii) that for any $z = (z_i)_{i=1}^{16} \in {}_0\mathbb{E}_{1,\nu}(I)$

$$\|\mathrm{tr}_{\partial\Omega}([K'(z_*)z]\nabla\theta_*)\|_{_0\mathbb{F}_\nu(I)} \le C \sum_{i=1}^{16} \|\mathrm{tr}_{\partial\Omega}(\partial_i K(z_*)\nabla\theta_*)\|_{_0\mathbb{F}_\nu(I)} \|\mathrm{tr}_{\partial\Omega} z_i\|_{_0\mathbb{F}_{1,\nu}(I)},$$

where the constant C is independent of the length of the interval $I \subset [0, T]$. By absolute continuity, for any given $\eta > 0$, there exists a partition $\{I_j : 0 \le j \le n-1\}$ of $[0, T]$ (which can be chosen to be compatible with the one for A') such that

$$\|\mathrm{tr}_{\partial\Omega}([K'(z_*)z]\nabla\theta_*)\|_{_0\mathbb{F}_\nu(I_j)} \le \eta \, \|z\|_{_0\mathbb{E}_{1,\nu}(I_j)}, \quad 0 \le j \le n-1,$$

for some constant C that is independent of the length of the interval I_j. See Proposition A.4 and [28, Theorems 4.2 and 4.5]. Finally, we note that the term $\mathsf{F}'(z_*(t))z$ is of lower order in z and can therefore be handled by a standard perturbation argument. The assertion follows now by similar arguments as in the proof of Proposition 4.2. □

5. Local well-posedness

Theorem 5.1. (Local existence and uniqueness for the abstract problem) *Assume* (2.1) *and* (3.2).

(a) *Let* $\mathcal{M}_\mu = \{z \in X_{\gamma,\mu} : \mathsf{B}(z)z = 0\}$. *Then for every* $z_0 \in \mathcal{M}_\mu$, *there exists* $T > 0$ *such that the nonlinear system* (3.7) *has a unique solution* $z \in \mathbb{E}_{1,\mu}(J_T)$. *The solution can be continued to a maximal solution* $z = z(\cdot, z_0)$ *on an interval* $[0, T_+(z_0))$.

(b) *Let* $T < T_+(z_0)$. *Then, there exists a number* $\rho > 0$ *such that the system* (3.7) *has a unique solution* $z(\cdot, w_0) \in \mathbb{E}_{1,\mu}(J_T)$ *for each initial value* $w_0 \in \mathcal{M}_\mu \cap B_{X_{\gamma,\mu}}(z_0, \rho)$. *Moreover, the mapping*

$$[w_0 \mapsto z(\cdot, w_0)] : \mathcal{M}_\mu \cap B_{X_{\gamma,\mu}}(z_0, \rho) \to \mathbb{E}_{1,\mu}(J_T)$$

is locally Lipschitz continuous. Hence, (3.7) *generates a (locally) Lipschitz continuous semiflow on* $\mathcal{M}_\mu$.

(c) *Let* $T < T_+(z_0)$ *and* $z = z(\cdot, z_0)$ *be the (unique) solution of* (3.7). *Then,*

$$tz \in W^2_{p,\mu}(J_T; X_0) \cap W^1_{p,\mu}(J_T; X_1).$$

Moreover, $z \in C^1((0, T]; X_{\gamma,\mu})$.

Proof. In this proof, we will follow the ideas in [22, Theorem 14] and [26, Proposition 4.3.2].

(a) Fix $z_* \in \mathbb{E}_{1,\mu}(\mathbb{R}_+)$ with $z_*(0) = z_0$, whose existence is guaranteed by Lemma 3.1(b). We put

$$\mathsf{A}_*(t)z = \mathcal{A}'(z_*(t))z - \mathsf{F}'(z_*(t))z,$$
$$\mathsf{B}_*(t)z = \mathcal{B}'(z_*(t))z,$$

where the functions $(\mathcal{A}, \mathcal{B})$ are defined in (4.12), and consider the linear problem

$$\begin{cases} \partial_t z + \mathsf{A}_*(t)z = \mathcal{A}'(z_*)z_* - \mathcal{A}(z_*) + \mathsf{F}(z_*) - \mathsf{F}'(z_*)z_* & \text{in} \quad \Omega, \\ \mathsf{B}_*(t)z = \mathcal{B}'(z_*)z_* - \mathcal{B}(z_*) & \text{on} \quad \partial\Omega, \\ z(0) = z_0 & \text{in} \quad \Omega. \end{cases} \tag{5.1}$$

Note that the compatibility condition

$$\mathsf{B}_*(0)z_0 = \mathcal{B}'(z_0)z_0 - \mathcal{B}(z_0)$$

is satisfied, as $\mathcal{B}(z_0) = \mathsf{B}(z_0)z_0 = 0$ by assumption. Therefore, Proposition 4.3 implies that for any $T_0 > 0$, (5.1) has a unique solution $w \in \mathbb{E}_{1,\mu}(J_{T_0})$. Fix $T_0, R_0 > 0$. For every $T \in (0, T_0]$ and $R \in (0, R_0]$, we define a closed set in $\mathbb{E}_{1,\mu}(J_T)$ by

$$\Sigma(T, R) = \{z \in \mathbb{E}_{1,\mu}(J_T) : \|z - w\|_{\mathbb{E}_{1,\mu}(J_T)} \le R,\ \gamma_0 z = z_0\}.$$

Observe that, by [28, Theorems 4.2 and 4.5], there exists some $M > 0$ such that for all $T \in (0, T_0]$ and $R \in (0, R_0]$ and every $\hat{z} \in \Sigma(T, R)$, it holds that

$$\|\mathrm{tr}_{\partial\Omega}\hat{z}\|_{\mathbb{F}_\mu(J_T)}, \|\mathrm{tr}_{\partial\Omega}\nabla\hat{z}\|_{\mathbb{F}_\mu(J_T)}, \|\hat{z}\|_{\mathbb{E}_{1,\mu}(J_T)}, \|\hat{z}\|_{\mathbb{B}_\mu(J_T)} \le M. \tag{5.2}$$

Given any $\hat{z} \in \Sigma(T, R)$, we consider the linear problem

$$\begin{cases} \partial_t z + \mathsf{A}_*(t)z = \mathcal{A}'(z_*)\hat{z} - \mathcal{A}(\hat{z}) + \mathsf{F}(\hat{z}) - \mathsf{F}'(z_*)\hat{z} & \text{in} \quad \Omega, \\ \mathsf{B}_*(t)z = \mathcal{B}'(z_*)\hat{z} - \mathcal{B}(\hat{z}) & \text{on} \quad \partial\Omega, \\ z(0) = z_0 & \text{in} \quad \Omega. \end{cases} \tag{5.3}$$

As $\mathcal{B}(z_0) = \mathsf{B}(z_0)z_0 = 0$, the compatibility condition

$$\mathsf{B}_*(0)z_0 = \mathcal{B}'(z_0)z_0 - \mathcal{B}(z_0)$$

is satisfied, and we can infer from Proposition 4.3 that (5.3) has a unique solution $z = \mathcal{T}(\hat{z}) \in \mathbb{E}_{1,\mu}(J_T)$.

Then, it is clear that $z \in \Sigma(T, R)$ solves (3.7) iff it is a fixed point of $\mathcal{T}$ in $\Sigma(T, R)$. Note that $v = \mathcal{T}(\hat{z}) - w$ solves

$$\begin{cases} \partial_t v + \mathsf{A}_*(t)v = \mathbb{F}_*(\hat{z}) & \text{in} \quad \Omega, \\ \mathsf{B}_*(t)v = \mathbb{G}_*(\hat{z}) & \text{on} \quad \partial\Omega, \\ v(0) = 0 & \text{in} \quad \Omega, \end{cases}$$

where

$$\mathbb{F}_*(\hat{z}) = -\big(\mathcal{A}(\hat{z}) - \mathcal{A}(z_*) - \mathcal{A}'(z_*)(\hat{z}-z_*)\big) + \mathsf{F}(\hat{z}) - \mathsf{F}(z_*) - \mathsf{F}'(z_*)(\hat{z}-z_*),$$
$$\mathbb{G}_*(\hat{z}) = -\big(\mathcal{B}(\hat{z}) - \mathcal{B}(z_*) - \mathcal{B}'(z_*)(\hat{z}-z_*)\big).$$

In view of Proposition 4.3, there exits a constant $C > 0$, which is independent of $T \in (0, T_0]$, such that

$$\begin{aligned}\|\mathcal{T}(\hat{z}) - w\|_{\mathbb{E}_{1,\mu}(J_T)} \leq C(&\|\mathcal{A}(\hat{z}) - \mathcal{A}(z_*) - \mathcal{A}'(z_*)(\hat{z}-z_*)\|_{\mathbb{E}_{0,\mu}(J_T)}\\ &+ \|\mathsf{F}(\hat{z}) - \mathsf{F}(z_*) - \mathsf{F}'(z_*)(\hat{z}-z_*)\|_{\mathbb{E}_{0,\mu}(J_T)}\\ &+ \|\mathcal{B}(\hat{z}) - \mathcal{B}(z_*) - \mathcal{B}'(z_*)(\hat{z}-z_*)\|_{\mathbb{F}_{\mu}(J_T)}),\end{aligned}$$

where we have used the fact that

$$\mathcal{B}(\hat{z}) - \mathcal{B}(z_*) - \mathcal{B}'(z_*)(\hat{z}-z_*) \in {}_0\mathbb{F}_{\mu}(J_T).$$

Using (5.2) and (B.5), one verifies that $\|\mathcal{T}(\hat{z}) - w\|_{\mathbb{E}_{1,\mu}(J_T)} \leq R$, provided T and R are chosen small enough. This shows that $\mathcal{T}$ maps $\Sigma(T, R)$ into itself. To show that $\mathcal{T} : \Sigma(T, R) \to \Sigma(T, R)$ is a strict contraction, we pick functions $\hat{z}, \bar{z} \in \Sigma(T, R)$. Then, we obtain

$$\begin{aligned}\|\mathcal{T}(\hat{z}) - \mathcal{T}(\bar{z})\|_{\mathbb{E}_{1,\mu}(J_T)} \leq C(&\|\mathcal{A}(\hat{z}) - \mathcal{A}(\bar{z}) - \mathcal{A}'(z_*)(\hat{z}-\bar{z})\|_{\mathbb{E}_{0,\mu}(J_T)}\\ &+ \|\mathsf{F}(\hat{z}) - \mathsf{F}(\bar{z}) - \mathsf{F}'(z_*)(\hat{z}-\bar{z})\|_{\mathbb{E}_{0,\mu}(J_T)}\\ &+ \|\mathcal{B}(\hat{z}) - \mathcal{B}(\bar{z}) - \mathcal{B}'(z_*)(\hat{z}-\bar{z})\|_{\mathbb{F}_{\mu}(J_T)})\end{aligned}$$

for some constant C that is independent of $T \in (0, T_0]$. Employing (5.2) and (B.5), (B.6), one verifies that

$$\|\mathcal{T}(\hat{z}) - \mathcal{T}(\bar{z})\|_{\mathbb{E}_{1,\mu}(J_T)} \leq \frac{1}{2}\|\bar{z} - \hat{z}\|_{\mathbb{E}_{1,\mu}(J_T)},$$

provided T and R are chosen sufficiently small.

The contraction mapping principle implies the existence of a unique solution $z \in \Sigma(T, R)$ to (3.7) on the time interval $[0, T]$. A standard argument then yields that z is also the unique solution in $\mathbb{E}_{1,\mu}(J_T)$.

The existence of a maximal interval of existence $[0, T_+(z_0))$ can be obtained in a standard way as in [32, Corollary 5.1.2].

(b) Pick an arbitrary $T \in (0, T_+(z_0))$ and let $z = z(\cdot, z_0)$ be the (unique) solution of (3.7) obtained in part (a). Then, $w \in \mathbb{E}_{1,\mu}(J_T)$ is a solution of (3.7) with initial value $w_0 \in \mathcal{M}_\mu$ iff $w = z + v$, where $v \in \mathbb{E}_{1,\mu}(J_T)$ solves the system

$$\begin{cases} \partial_t v + \mathsf{A}_0(t)v = \mathbb{F}(v(t)) & \text{in} \quad \Omega,\\ \mathsf{B}_0(t)v = \mathbb{G}(v(t)) & \text{on} \quad \partial\Omega,\\ v(0) = v_0 = w_0 - z_0 & \text{in} \quad \Omega \end{cases} \tag{5.4}$$

on $[0, T]$, with

$$\begin{aligned}
\mathsf{A}_0(t)v &= \mathcal{A}'(z(t))v - \mathsf{F}'(z(t))v = \mathsf{A}(z(t))v + [\mathsf{A}'(z(t))v]z(t) - \mathsf{F}'(z(t))v,\\
\mathsf{B}_0(t)v &= \mathcal{B}'(z(t))v = \mathsf{B}(z(t))v + [\mathsf{B}'(z(t))v]z(t),\\
\mathbb{F}(v(t)) &= -\big(\mathcal{A}(z(t)+v(t)) - \mathcal{A}(z(t)) - \mathcal{A}'(z(t))v(t)\big) + \mathsf{F}(z(t)+v(t)) - \mathsf{F}(z(t))\\
&\quad - \mathsf{F}'(z(t))v(t),\\
\mathbb{G}(v(t)) &= -\big(\mathcal{B}(z(t)+v(t)) - \mathcal{B}(z(t)) - \mathcal{B}'(z(t))v(t)\big).
\end{aligned}$$

It follows from Proposition B.3 that

$$\mathbb{F} \in C^1(\mathbb{E}_{1,\mu}(J_T), \mathbb{E}_{0,\mu}(J_T)) \quad \text{and} \quad \mathbb{G} \in C^1(\mathbb{E}_{1,\mu}(J_T), \mathbb{F}_\mu(J_T)). \tag{5.5}$$

Easy computations show that

$$\mathbb{F}(0) = 0, \qquad \mathbb{G}(0) = 0, \quad \mathbb{F}'(0) = 0, \qquad \mathbb{G}'(0) = 0. \tag{5.6}$$

Note that the compatibility condition

$$\mathsf{B}_0(0)v(0) := \mathcal{B}'(z_0)v(0) = \mathbb{G}(v(0)) \tag{5.7}$$

is satisfied, as $\mathcal{B}(z_0) = \mathcal{B}(w_0) = 0$ by assumption. Let

$$X^0_{\gamma,\mu} := \{\hat{z}_0 \in X_{\gamma,\mu} : \mathcal{B}'(z_0)\hat{z}_0 = 0\}.$$

We then introduce the map $\mathcal{F} : X^0_{\gamma,\mu} \times \mathbb{E}_{1,\mu}(J_T) \to \mathbb{E}_{1,\mu}(J_T)$, defined by

$$\mathcal{F}(\hat{z}_0, \hat{v}) = \hat{v} - \mathcal{S}(\mathbb{F}(\hat{v}), \mathbb{G}(\hat{v}), \hat{z}_0 + \mathcal{R}(z_0)\mathbb{G}(\hat{v}(0))),$$

where $\mathcal{S}$ is the solution operator defined in Proposition 4.3 and $\mathcal{R}(z_0) : Y_{\gamma,\mu} \to X_{\gamma,\mu}$ is the bounded right inverse of $\mathcal{B}'(z_0)$ asserted by Lemma B.3. Observe that $\mathcal{F}(0,0) = 0$ and the compatibility condition

$$\mathcal{B}'(z_0)\left(\hat{z}_0 + \mathcal{R}(z_0)\mathbb{G}(\hat{v}(0))\right) = \mathbb{G}(\hat{v}(0))$$

is satisfied for each $v \in \mathbb{E}_{1,\mu}(J_T)$. It follows from (5.5) that

$$\mathcal{F} \in C^1(X^0_{\gamma,\mu} \times \mathbb{E}_{1,\mu}(J_T), \mathbb{E}_{1,\mu}(J_T))$$

and from (5.6) that $\partial_2\mathcal{F}(0,0) \in \mathcal{L}is(\mathbb{E}_{1,\mu}(J_T))$. The implicit function theorem then implies that there exist some $r > 0$ and $\Psi \in C^1(B_{X^0_{\gamma,\mu}}(0,r), \mathbb{E}_{1,\mu}(J_T))$ such that $\hat{v} = \Psi(\hat{z}_0)$ with $\hat{z}_0 \in B_{X^0_{\gamma,\mu}}(0,r)$ iff $\mathcal{F}(\hat{z}_0, \hat{v}) = 0$, or equivalently, $\hat{v}$ solves

$$\begin{cases}
\partial_t\hat{v} + \mathsf{A}_0(t)\hat{v} = \mathbb{F}(\hat{v}(t)) & \text{in} \quad \Omega,\\
\mathsf{B}_0(t)\hat{v} = \mathbb{G}(\hat{v}(t)) & \text{on} \quad \partial\Omega,\\
\hat{v}(0) = \hat{z}_0 + \mathcal{R}(z_0)\mathbb{G}(\hat{v}(0)) & \text{in} \quad \Omega.
\end{cases}$$

We define $\mathcal{P}(z_0) : X_{\gamma,\mu} \to X^0_{\gamma,\mu}$ by $\mathcal{P}(z_0)\widetilde{z} = (I - \mathcal{R}(z_0)\mathcal{B}'(z_0))\widetilde{z}$. For sufficiently small $\rho > 0$, and $w_0 \in \mathcal{M}_\mu \cap \in B_{X_{\gamma,\mu}}(z_0, \rho)$, we choose

$$\hat{z}_0 = \mathcal{P}(z_0)v_0 \in B_{X^0_{\gamma,\mu}}(0, r), \quad \text{where } v_0 = w_0 - z_0.$$

In view of (5.7), it holds that

$$v_0 = \hat{z}_0 + \mathcal{R}(z_0)\mathcal{B}'(z_0)v_0 = \hat{z}_0 + \mathcal{R}(z_0)\mathbb{G}(v(0)).$$

Therefore, v and $\hat{v}$ solve the same system of equations. We conclude that $v := \Psi(\mathcal{P}(z_0)(w_0 - z_0))$ is the unique solution of (5.4) on $[0, T]$ with initial value v_0. Hence, $w = z + \Psi(\mathcal{P}(z_0)(w_0 - z_0))$ is the (unique) solution to (3.7) with initial value w_0 on $[0, T]$. Setting $z(\cdot, w_0) = z(\cdot, z_0) + \Psi(\mathcal{P}(z_0)(w_0 - z_0))$, we can infer that the mapping

$$[w_0 \mapsto z(\cdot, w_0)] : \mathcal{M}_\mu \cap B_{X_{\gamma,\mu}}(z_0, \rho) \to \mathbb{E}_{1,\mu}(J_T)$$

is Lipschitz continuous.

(c) Fix $T \in (0, T_+(z_0))$ and $\epsilon \in (0, 1)$ so small that $(1 + \epsilon)T < T_+(z_0)$. Let $z \in \mathbb{E}_{1,\mu}(J_T)$ be the unique solution of (3.7) with initial value z_0. Let $z_\lambda(t) = z(\lambda t)$. Then, $v = z_\lambda$ solves

$$\begin{cases} \partial_t v + \mathsf{A}_0(t)v = \mathfrak{F}(\lambda, v(t)) & \text{in} \quad \Omega, \\ \mathsf{B}_0(t)v = \mathfrak{G}(v(t)) & \text{on} \quad \partial\Omega, \\ v(0) = z_0 & \text{in} \quad \Omega \end{cases} \tag{5.8}$$

on $[0, T]$, where A_0 and B_0 are defined as in (5.4) and

$$\mathfrak{F}(\lambda, v(t)) = -\big(\lambda\mathcal{A}(v(t))) - \mathcal{A}'(z(t))v(t)\big) + \lambda\mathsf{F}(v(t)) - \mathsf{F}'(z(t))v(t),$$
$$\mathfrak{G}(v(t)) = -\big(\mathcal{B}(v(t)) - \mathcal{B}'(z(t)v(t)\big),$$

with $(\mathcal{A}, \mathcal{B})$ defined in (4.12). As $\mathfrak{F}(1, v) = -\big(\mathcal{A}(v) - \mathcal{A}'(z)v\big) + \mathsf{F}(v) - \mathsf{F}'(z)v$, one readily verifies that

$$\partial_2\mathfrak{F}(1, z) = 0, \quad \mathfrak{G}'(z) = 0.$$

Similar to part (b), we define $\mathcal{F}_0 : (1 - \epsilon, 1 + \epsilon) \times \mathbb{E}_{1,\mu}(J_T) \to \mathbb{E}_{1,\mu}(J_T)$ by

$$\mathcal{F}_0(\lambda, v) = v - \mathcal{S}(\mathfrak{F}(\lambda, v), \mathfrak{G}(v), \hat{z}_0 + \mathcal{R}(z_0)\mathfrak{G}(v(0))),$$

where $\hat{z}_0 = z_0 - \mathcal{R}(z_0)\mathcal{B}'(z_0)z_0 \in X^0_{\gamma,\mu}$. Note that the compatibility condition

$$\mathcal{B}'(z_0)(\hat{z}_0 + \mathcal{R}(z_0)\mathfrak{G}(v(0))) = \mathfrak{G}(v(0))$$

is again satisfied. We have $\mathcal{F}_0(1, z) = 0$ and $\partial_2\mathcal{F}_0(1, z) \in \mathcal{L}is(\mathbb{E}_{1,\mu}(J_T))$. Therefore, the implicit function theorem implies that there exist $\delta \in (0, \epsilon)$ and $\Psi_0 \in C^1((1 - \delta, 1 + \delta), \mathbb{E}_{1,\mu}(J_T))$ such that $v = \Psi_0(\lambda)$ iff v solves

$$\begin{cases} \partial_t v + \mathsf{A}_0(t)v = \mathfrak{F}(\lambda, v(t)) & \text{in } \Omega, \\ sB_0(t)v = \mathfrak{G}(v(t)) & \text{on} \partial\Omega, \\ v(0) = z_0 - \mathcal{R}(z_0)\mathcal{B}'(z_0)z_0 + \mathcal{R}(z_0)\mathfrak{G}(v(0)) & \text{in } \Omega. \end{cases}$$

From $\mathcal{F}_0(1, z) = 0$, we infer that $z = \Psi_0(1)$. We want to show that $\Psi_0(\lambda) = z_\lambda$. To this end, notice that $z_0(\lambda) := \gamma_0 \Psi_0(\lambda)$ satisfies

$$\begin{aligned} z_0(\lambda) &= z_0 - \mathcal{R}(z_0)\mathcal{B}'(z_0)z_0 + \mathcal{R}(z_0)\mathfrak{G}(z_0(\lambda)) \\ &= z_0 - \mathcal{R}(z_0)\big(\mathcal{B}(z_0(\lambda) - \mathcal{B}'(z_0)(z_0(\lambda) - z_0)\big). \end{aligned}$$

By using the fact that $\mathcal{B}(z_0) = \mathsf{B}(z_0)z_0 = 0$, we further obtain

$$z_0(\lambda) - z_0 = -\mathcal{R}(z_0)\big(\mathcal{B}(z_0(\lambda)) - \mathcal{B}(z_0) - \mathcal{B}'(z_0)(z_0(\lambda) - z_0))\big).$$

We can thus conclude that

$$\begin{aligned} \|z_0(\lambda) - z_0\|_{X_{\gamma,\mu}} &\le \Phi(\|z_0(\lambda) - z_0\|_{X_{\gamma,\mu}})\|z_0(\lambda) - z_0\|_{X_{\gamma,\mu}} \\ &\le \Phi(\|\Psi_0(\lambda) - \Psi_0(1)\|_{\mathbb{E}_{1,\mu}(J_T)})\|z_0(\lambda) - z_0\|_{X_{\gamma,\mu}}. \end{aligned}$$

By choosing δ so small that

$$\sup_{\lambda \in (1-\delta, 1+\delta)} \Phi(\|\Psi_0(\lambda) - \Psi_0(1)\|_{\mathbb{E}_{1,\mu}(J_T)}) \le 1/2,$$

we have $z_0(\lambda) = z_0$ for all $\lambda \in (1-\delta, 1+\delta)$. Thus, $\Psi_0(\lambda)$ solves (5.8) and hence $\Psi_0(\lambda) = z_\lambda$. The differentiability of Ψ_0 implies that $\Psi_0'(1) = t\partial_t z \in \mathbb{E}_{1,\mu}(J_T)$. Therefore, $\partial_t(tz) = z + t\partial_t z \in \mathbb{E}_{1,\mu}(J_T)$. Then, the asserted the regularity of z follows. □

In the following, we will discuss the remaining issues concerning the pressure function π and the constraint $|m| = 1$.

Proposition 5.2. *Given $T > 0$, the following statements are equivalent:*

(a) (3.1) *has a solution* $(u, F, \theta, m, \pi) \in \mathbb{E}_{1,\mu}(J_T) \times L_{p,\mu}(J_T; \dot{H}^1_p(\Omega))$.

(b) (3.7) *has a solution* $(u, F, \theta, m) \in \mathbb{E}_{1,\mu}(J_T)$.

Proof. The implication (a)⇒(b) follows by just applying P_H to both sides of the u equation in (3.1). We are left to show (b)⇒(a). Suppose $z = (u, F, \theta, m) \in \mathbb{E}_{1,\mu}(J_T)$ solves (3.7). Let

$$v = -u \cdot \nabla u + \nabla \cdot (\upmu(\theta)\nabla u) - \nabla \cdot (\nabla m \odot \nabla m) + \nabla \cdot (FF^\top).$$

Let π be an W^1_p solution to $\Delta\pi = \nabla \cdot v$ in Ω with $\partial_\nu \pi = v \cdot \nu$ on $\partial\Omega$, i.e.,

$$(\nabla\pi(t)|\nabla\phi) = (v(t)|\nabla\phi), \qquad \forall \phi \in \dot{H}^1_{p'}(\Omega), \quad p' = p/(p-1).$$

From standard elliptic theory we conclude that $\pi \in L_{p,\mu}(J_T; \dot{H}^1_p(\Omega))$ and $P_H v(t) = v(t) - \nabla\pi(t)$. Then by (3.7), we have

$$\partial_t u + \nabla\pi - v = \partial_t u - P_H v = 0.$$

Hence, (u, F, θ, m, π) solves $(3.1)_1$. □

Now, we are in a position to state the main theorem concerning local well-posedness of (3.1). To this end, we define the state manifold of (3.1) by

$$\mathcal{SM}_\mu := \{z = (u, F, \theta, m) \in X_{\gamma,\mu} : \ \theta > 0, \ |m| = 1, \ \mathsf{B}(z)z = 0\}.$$

Theorem 5.3. (Local well-posedness of (3.1)) *Assume* (2.1) *and* (3.2).

(a) *Suppose that* $z_0 = (u_0, F_0, \theta_0, m_0) \in \mathcal{M}_\mu$. *Then, there exists a number* $T > 0$ *such that* (3.1) *has a unique solution*

$$\widetilde{z}(\cdot, z_0) = (u, F, \theta, m, \pi) \in \mathbb{E}_{1,\mu}(J_T) \times L_{p,\mu}(J_T; \dot{H}^1_q(\Omega)).$$

Each solution can be extended to a maximal existence interval $[0, T_+(z_0))$. *If, in addition,* $|m_0| \equiv 1$, *then the solution also satisfies*

$$|m(t)| \equiv 1, \quad t \in [0, T_+(z_0)).$$

Moreover, it holds that

$$\theta(t, x) \geq \min_{\overline{\Omega}} \theta_0(x), \quad (t, x) \in [0, T_+(z_0)) \times \overline{\Omega}.$$

(b) *Let* $T < T_+(z_0)$. *Then, there exists a number* $\rho > 0$ *such that for every* $w_0 \in \mathcal{SM}_\mu \cap B_{X_{\gamma,\mu}}(z_0, \rho)$, *the unique solution* $\widetilde{z}(\cdot, w_0)$ *of* (3.1) *with initial condition* w_0 *belongs to* $\mathbb{E}_{1,\mu}(J_T) \times L_{p,\mu}(J_T; \dot{H}^1_q(\Omega))$. *Moreover, the mapping*

$$[w_0 \mapsto \widetilde{z}(\cdot, w_0)] : \mathcal{SM}_\mu \cap B_{X_{\gamma,\mu}}(z_0, \rho) \to \mathbb{E}_{1,\mu}(J_T) \times L_{p,\mu}(J_T; \dot{H}^1_q(\Omega))$$

is locally Lipschitz continuous. Hence, the system (3.1) *generates a (Lipschitz) continuous semiflow on* $\mathcal{SM}_\mu$.

Proof. (a) The fact that $|m(t)| = 1$ up to $T_+(z_0)$, provided $|m_0| = 1$, follows from a parabolic maximum principle (c.f. [10, Theorem 2.5]). By (3.3), $\theta_0 \in C^1(\overline{\Omega})$, so that $\min_{\overline{\Omega}} \theta_0(x)$ exists. For $(t, x) \in [0, T_+(z_0)) \times \Omega$, let $\psi(t, x) = \min_{\overline{\Omega}} \theta_0(x) - \theta(t, x)$. Then, we can derive from (3.1) that ψ solves

$$\begin{aligned} \partial_t \psi + u \cdot \nabla \psi \leq \nabla \cdot (K(z) \nabla \psi) \quad & \text{in} \quad \Omega, \\ \nu \cdot \mathrm{tr}_{\partial\Omega}(K(z)\nabla\psi) = 0 \quad & \text{on} \quad \partial\Omega, \\ \psi(0, x) \leq 0 \quad & \text{in} \quad \Omega. \end{aligned} \tag{5.9}$$

Multiplying both sides of $(5.9)_1$ by $\psi_+ = \max\{\psi, 0\}$ and integrating over Ω, we can show that

$$\partial_t \left(\frac{\|\psi_+\|_2^2}{2} \right) + c\|\nabla\psi_+\|_2^2 \leq \partial_t \left(\frac{\|\psi_+\|_2^2}{2} \right) + (K(z)\nabla\psi_+ | \nabla\psi_+)_\Omega \leq 0. \tag{5.10}$$

In fact, we can compute

$$(\partial_t \psi | \psi_+)_\Omega = (\partial_t \psi_+ | \psi_+)_\Omega = \partial_t \frac{\|\psi_+\|_2^2}{2},$$

$$(u \cdot \nabla \psi | \psi_+)_\Omega = (u \cdot \nabla \psi_+ | \psi_+)_\Omega = \int_\Omega u \cdot \nabla \frac{|\psi_+|^2}{2} \, \mathrm{d}x = 0,$$

$$(\nabla \cdot (K(z) \nabla \psi) | \psi_+)_\Omega = -\int_\Omega (K(z) \nabla \psi) \cdot \nabla \psi_+ \, \mathrm{d}x = -(K(z) \nabla \psi_+ | \nabla \psi_+)_\Omega.$$

Integrating (5.10) with respect to t, and using the fact that $\psi_+(0, x) = 0$, we conclude that $\|\psi_+(t)\|_2^2 = 0$ for $t \in [0, T_+(z_0))$. Hence, $\psi_+(t, x) = 0$ for $(t, x) \in [0, T_+(z_0)) \times \overline{\Omega}$.

(b) This part follows directly from Theorem 5.1(b), the proof of Proposition 5.2 and the fact that

$$[z \mapsto (-u \cdot \nabla u + \nabla \cdot (\mu(\theta) \nabla u) - \nabla \cdot (\nabla m \odot \nabla m) + \nabla \cdot (F F^\top))] \in C^1(\mathbb{E}_{1,\mu}(J_T), L_{p,\mu}(J_T; L_p(\Omega; \mathbb{R}^3))).$$

□

6. Stability and long-time behavior

In this section, we will study global existence and stability of solutions to (1.1). The next theorem establishes the long-time behavior of solutions.

Theorem 6.1. *Assume* (2.1), (3.2), $|m_0| = 1$, *and the positivity condition* $\theta_0 > 0$. *Let* $z = z(\cdot, z_0)$ *be the solution of* (1.1)*, defined on its maximal interval of existence* $[0, T_+(z_0))$. *Then, the following properties hold.*

(a) *We have the following alternatives:*

(i) $T_+(z_0) = \infty$, *that is, z is a global solution;*

(ii) $\lim_{t \to T_+(z_0)} z(t)$ *does not exist in* $X_{\gamma,\mu}$.

(b) *Suppose*

$$\sup_{t \in [\delta, T_+(z_0))} \|z(t)\|_{X_{\gamma,\overline{\mu}}} < \infty \quad \text{for some } \delta \in (0, T_+(z_0)) \text{ and some } \overline{\mu} \in (\mu, 1].$$

Then, z exists globally and $\mathrm{dist}(z(t), \mathcal{E}) \to 0$ *in* $X_{\gamma,1}$ *as* $t \to \infty$.

Proof. (a) We will prove the assertion by following the strategy in [32, Corollary 5.1.2]. Assume that $T_+(z_0) < \infty$ and $z(\cdot, z_0)$ converges to some z_1 in $X_{\gamma,\mu}$ as $t \to T_+(z_0)$. Lemma B.1 implies that $\mathsf{B}(z_1) z_1 = 0$. Combining Theorem 5.3(a) and the assumption, we have that there exists an $\eta > 0$ depending on $\min_{\overline{\Omega}} \theta_0$ such that

$$\mathrm{dist}_{X_{\gamma,\mu}}(z(t), \partial V_\mu) \geq \eta, \quad \text{for all } t \in [0, T_+(z_0)), \tag{6.1}$$

where $V_\mu = \{z = (u, F, \theta, m) \in X_{\gamma,\mu} : \theta > 0\}$. We thus infer that $z_1 \in \mathcal{SM}_\mu$. Then, the orbit $\mathcal{V} := \{z(t) : 0 \leq t < T_+(z_0)\}$ is relatively compact in $\mathcal{SM}_\mu$.

It follows from Theorem 5.3(b) and a compactness argument that there exists $T_0 > 0$ such that for each $s \in [0, T_+(z_0))$, system (3.1) with initial value $z(s)$ has a unique solution in $\mathbb{E}_{1,\mu}(J_{T_0})$. Fixing $s_0 \in (T_+(z_0) - T_0, T_+(z_0))$, system (3.1) with initial value $z(s_0)$ has a solution $v \in \mathbb{E}_{1,1}(J_{T_0})$, which, by uniqueness, coincides with $z(s_0 + \cdot, z_0)$ on $[s_0, T_+(z_0))$. In view of Proposition 5.2, the solution $\widetilde{z}(\cdot, z_0)$ of (1.1) can be extended beyond $T_+(z_0)$, a contradiction.

(b) We will prove the assertion by following the strategy in [32, Section 5.7]. By Theorem 5.3(b), the system (3.1) defines a local semiflow on $\mathcal{SM}_\mu$. From the assumption and the compact embedding

$$X_{\gamma,\overline{\mu}} \hookrightarrow X_{\gamma,\mu},$$

we infer that the orbit $\mathcal{V} := \{z(t) : 0 \le t < T^+(z_0)\}$ is relatively compact in $\mathcal{SM}_\mu$. Denote the closure of $\mathcal{V}$ in $X_{\gamma,\mu}$ by $\overline{\mathcal{V}}$. It follows from a similar argument as in part (a) that there exist a number $T_0 > 0$ and an open neighborhood $\mathcal{U}$ of $\overline{\mathcal{V}}$ in $\mathcal{M}_\mu$ such that for every $\widetilde{z}_0 \in \mathcal{U}$, (3.7) admits a unique solution $\widetilde{z} \in \mathbb{E}_{1,\mu}(J_{T_0})$. Moreover, the solution map $G_1 : \mathcal{U} \to \mathbb{E}_{1,\mu}(J_{T_0})$ is continuous. This implies that for any $t \in [0, T_+(z_0))$, the solution of (3.7) with initial condition $z(t)$ exists on the interval $[t, t + T_0]$, which further shows that $T_+(z_0) = \infty$. Now it follows from Proposition 5.2 that the solution to (1.1) is global.

As above, one sees that (3.1) also defines a local semiflow on $\mathcal{SM}_1$, equipped with the metric induced by $X_{\gamma,1}$. It follows from the inequality

$$\begin{aligned}\|z(T_0)\|_{X_{\gamma,1}} &\le \|z\|_{C([T_0/2,T_0];X_{\gamma,1})} \le C(T_0)\|z\|_{\mathbb{E}_{1,1}([T_0/2,T_0])} \\ &\le C(T_0)(T_0/2)^{\mu-1}\|z\|_{\mathbb{E}_{1,\mu}(J_{T_0})}\end{aligned}$$

that the map $G_2 : \mathbb{E}_{1,\mu}(J_{T_0}) \to X_{\gamma,1} : z \mapsto z(T_0)$ is continuous. This implies that the composition map $G = G_2 \circ G_1 : \mathcal{U} \to X_{\gamma,1} : z \mapsto G_1(z)(T_0)$ is continuous. We thus infer that the orbit $\{z(t)\}_{t \ge T_0}$ is relatively compact in $\mathcal{SM}_1$ because the continuous image of a relatively compact set is again relatively compact. Recall that the definition of ω-limit set of (3.7) is given by

$$\omega(z_0) := \{w \in X_{\gamma,1} : \exists t_n \to \infty \text{ s.t. } \|z(t_n) - w\|_{X_{\gamma,1}} = 0 \text{ as } n \to \infty\}.$$

By [1, Theorem 17.2], $\omega(z_0)$ is nonempty, compact, connected in $\mathcal{SM}_1$ and

$$\lim_{t\to\infty} \operatorname{dist}_{X_{\gamma,1}}(z(t), \omega(z_0)) = 0. \tag{6.2}$$

Now following a similar computation as in [10, Proposition 4.1], we can show that $-\mathsf{N}$ is a strict Lyapunov functional for (3.1). Therefore, $\omega(z_0) \subset \mathcal{E}$. Combining with (6.2), this implies

$$\lim_{t\to\infty} \operatorname{dist}_{X_{\gamma,1}}(z(t), \mathcal{E}) = 0.$$

□

Our last result is about the qualitative behavior of solutions near constant equilibria. Consider the set of constant equilibria of (3.7):

$$\mathcal{E}_c := \{0\} \times \{0_3\} \times \mathbb{R}_+ \times \mathbb{R}^3$$

which is a subset of $\mathcal{E}$, the set of equilibria. Let $z_* \in \mathcal{E}_c$ be given. Then, one readily verifies that

$$([\mathsf{A}'(z_*)z]z_*, \mathsf{F}'(z_*)z, [\mathsf{B}'(z_*)z]z_*) = (0, 0, 0), \quad z \in X_1. \tag{6.3}$$

Therefore, the linearization of (3.7) at $z_* \in \mathcal{E}_c$ is given by

$$\begin{aligned}
\mathsf{A}_* z &:= \mathsf{A}(z_*)z \\
&= \begin{bmatrix} -\mu(\theta_*)P_H\Delta u & 0 & 0 & 0 \\ 0 & -\kappa(\theta_*)\Delta F & 0 & 0 \\ 0 & 0 & -K(z_*) : \nabla^2\theta & 0 \\ 0 & 0 & 0 & (\beta(\theta_*)\mathsf{M}(m_*) - \alpha(\theta_*)I_3)\Delta m \end{bmatrix}, \\
\mathsf{B}_* z &= \nu \cdot \mathrm{tr}_{\partial\Omega}(K(z_*)\nabla\theta), \qquad z = (u, F, \theta, m).
\end{aligned}$$

Note that $(\mathsf{A}_*, \mathsf{B}_*) \in \mathcal{L}(X_1, X_0 \times Y_1)$, where $Y_1 = W_p^{1-1/p}(\partial\Omega)$. Hence, $A_0 := \mathsf{A}_*|_{N(\mathsf{B}_*)}$ is well-defined, where $N(\mathsf{B}_*)$ is the null space of B_*.
The next result will be important for proving stability of constant equilibria.

Proposition 6.2. *Each constant equilibrium $z_* \in \mathcal{E}_c$ is normally stable.*

Proof. By definition of normal stability, we need to show that

(i) near z_*, $\mathcal{E}_c$ is a C^1-manifold in X_1 of finite dimension,
(ii) the tangent space of $\mathcal{E}_c$ at z_* is isomorphic to $N(A_0)$,
(iii) 0 is a semi-simple eigenvalue of A_0, i.e., $X_0 = N(A_0) \oplus R(A_0)$,
(iv) $\sigma(A_0) \setminus \{0\} \subset \{z \in \mathbb{C} : \mathrm{Re} z > 0\}$.

We immediately see that (i) is satisfied, as $\mathcal{E}_c$ is a linear space of dimension 4.

Suppose $z = (u, F, \theta, m)$ is an eigenvector of A_0 subject to an eigenvalue $\lambda \in \mathbb{C}$, i.e., $A_0 z = \lambda z$. In other words, $\mathsf{B}_* z = 0$ on $\partial\Omega$, and $\mathsf{A}_* z = \lambda z$ in Ω. Taking the inner product of the later identity with $\bar{z}$ and using integration by parts, we can derive

$$\begin{aligned}
\mathrm{Re}\,\lambda \|z\|_2^2 &= \mu(\theta_*)\|\nabla u\|_2^2 + \kappa(\theta_*)\|\nabla F\|_2^2 + \mathrm{Re}\,(K(z_*)\nabla\theta|\nabla\bar{\theta})_\Omega + \alpha(\theta_*)\|\nabla m\|_2^2 \\
&\geq \underline{\mu}\|\nabla u\|_2^2 + \underline{\kappa}\|\nabla F\|_2^2 + c\|\nabla\theta\|_2^2 + \underline{\alpha}\|\nabla m\|_2^2,
\end{aligned}$$

where we use assumption (2.1) and the fact that $\mathrm{Re}\,(\mathsf{M}(m_*)\Delta m|\overline{m})_\Omega = 0$ (see [10, Section 3]). Hence, $\mathrm{Re}\,\lambda \geq 0$. Furthermore, when $\mathrm{Re}\,\lambda = 0$, we get that $z \in \{0\} \times \{0_3\} \times \mathbb{R} \times \mathbb{R}^3$, thus $\sigma(A_0) \cap i\mathbb{R} = \{0\}$ and $N(A_0) = \{0\} \times \{0_3\} \times \mathbb{R} \times \mathbb{R}^3$. This shows that (iii) and (iv) hold true.

Finally, we show that 0 is a semi-simple eigenvalue. Since A_0 has compact resolvent, it suffices to show that $N(A_0) = N(A_0^2)$. Since $N(A_0) \subset N(A_0^2)$, we just need to show

$N(A_0^2) \subset N(A_0)$. For $w = (v, J, \vartheta, n) \in N(A_0^2)$, let $z = (0, 0, \theta, m) \in N(A_0)$ such that $A_0 w = z$. Then, we can compute

$$\|z\|_2^2 = (\mathsf{A}_0 w|z)_\Omega = (-K(z_*) : \nabla^2 \vartheta|\theta)_\Omega + ((\beta(\theta_*)\mathsf{M}(m_*) - \alpha(\theta_*)I_3)\Delta n|m)_\Omega = 0,$$

where we use the fact that θ, m are constants in $N(A_0)$ and $\mathsf{B}_* w = 0$ on $\partial\Omega$. Hence, $A_0 w = z = 0$ and $w \in N(A_0)$. This yields that 0 is a semi-simple eigenvalue.

Finally, it follows from [33, Remark 2.2] that all equilibria near z_* are contained in a C^1 manifold of dimension 4. □

By adapting the proof of *the generalized principle of linearized stability* provided in [32, Section 5.3], we can obtain the following stability property of $\mathcal{E}_c$.

Theorem 6.3. *Assume* (2.1) *and* (3.2)*. Then, each equilibrium $z_* \in \mathcal{E}_c$ is stable in $X_{\gamma,\mu}$. Moreover, there exists $\delta > 0$ such that if $\|z_0 - z_*\|_{X_{\gamma,\mu}} \le \delta$, then the solution z of* (3.1) *with initial value z_0 exists globally and converges to some $z_\infty \in \mathcal{E}_c$ at an exponential rate in $X_{\gamma,1}$.*

Remark 6.4. We do not know of physical or mathematical principles that would help in characterizing the equilibrium state $z_\infty \in \mathcal{E}_c$.

Proof of Theorem 6.4 It will be convenient to center (3.7) around z_*, by setting $\bar{z} = z - z_*$. Then, (3.7) can be rewritten as

$$\begin{cases} \partial_t \bar{z} + \mathsf{A}_* \bar{z} = G(\bar{z}) & \text{in} \quad \Omega, \\ \mathsf{B}_* \bar{z} = H(\bar{z}) & \text{on} \quad \partial\Omega, \\ \bar{z}(0) = \bar{z}_0 = z_0 - z_* & \text{in} \quad \Omega, \end{cases} \tag{6.4}$$

where

$$\begin{aligned} G(\bar{z}) &= -\big(\mathsf{A}(z_* + \bar{z})(z_* + \bar{z}) - \mathsf{A}(z_*)\bar{z}\big) + \mathsf{F}(z_* + \bar{z}) \\ &= -\big((\mathsf{A}(z_* + \bar{z}) - \mathsf{A}(z_*))(z_* + \bar{z}) - [\mathsf{A}'(z_*)\bar{z}]z_*\big) + \mathsf{F}(z_* + \bar{z}) - \mathsf{F}(z_*) \\ &\quad - \mathsf{F}'(z_*)\bar{z} \\ H(\bar{z}) &= -\big(\mathsf{B}(z_* + \bar{z})(z_* + \bar{z}) - \mathsf{B}(z_*)\bar{z}\big) \\ &= -\big((\mathsf{B}(z_* + \bar{z}) - \mathsf{B}(z_*))(z_* + \bar{z}) - [\mathsf{B}'(z_*)z]z_*\big). \end{aligned}$$

Here, we used (6.3) and the relations $(\mathsf{A}(z_*)z_*, \mathsf{F}(z_*), \mathsf{B}(z_*)z_*) = (0, 0, 0)$ for the second line in the expressions of $G(\bar{z})$ and $H(\bar{z})$, respectively.

Theorem 5.3 shows that (6.4) has a unique solution $\bar{z}$ on some maximal interval of existence $[0, T_+)$.

In the following, we use the notation

$$X_0^c = N(A_0) = \{0\} \times \{0_3\} \times \mathbb{R}_+ \times \mathbb{R}^3, \quad X_0^s = R(A_0).$$

We know from Proposition 6.2 that $X_0 = X_0^c \oplus X_0^s$. Let P^c be the projection from X_0 onto X_0^c, and P^s the projection onto X_0^s. Then, we set $X_j^s = \mathsf{P}^s X_j$, $j \in \{0, 1, (\gamma, \mu)\}$.

We point out that $X_j^s = X_j \cap X_0^s$ and $\mathsf{P}^c X_j \doteq X^c$. Therefore, in the sequel, we will simply be using X^c, equipped with the norm induced by X_0. As X^c is finite dimensional, the projections P^c and P^s also provide the direct decomposition $X_j = X^c \oplus X_j^s$.

Following the arguments in parts (b) and (c) of the proof of [32, Theorem 5.3.1], near z_*, we decompose $\bar{z}$ as

$$\bar{z} = \mathsf{x} + \mathsf{y} := \mathsf{P}^c \bar{z} + \mathsf{P}^s \bar{z}.$$

Based on these notations, we define the normal form of (6.4) as

$$\begin{cases} \partial_t \mathsf{x} = T(\mathsf{x}, \mathsf{y}) & \text{in} \quad \Omega, \\ \partial_t \mathsf{y} + \mathsf{P}^s \mathsf{A}_* \mathsf{P}^s \mathsf{y} = R(\mathsf{x}, \mathsf{y}) & \text{in} \quad \Omega, \\ \mathsf{B}_* \mathsf{y} = S(\mathsf{x}, \mathsf{y}) & \text{on} \quad \partial\Omega, \\ \mathsf{x}(0) = \mathsf{x}_0, \quad \mathsf{y}(0) = \mathsf{y}_0 & \text{in} \quad \Omega. \end{cases} \tag{6.5}$$

Here, $\mathsf{x}_0 = \mathsf{P}^c \bar{z}_0$, $\mathsf{y}_0 = \mathsf{P}^s \bar{z}_0$ and

$$\begin{aligned} T(\mathsf{x}, \mathsf{y}) &= \mathsf{P}^c \left(G(\mathsf{x} + \mathsf{y}) - G(\mathsf{x}) \right) - \mathsf{P}^c \mathsf{A}_* \mathsf{y}, \\ R(\mathsf{x}, \mathsf{y}) &= \mathsf{P}^s \left(G(\mathsf{x} + \mathsf{y}) - G(\mathsf{x}) \right), \\ S(\mathsf{x}, \mathsf{y}) &= H(\mathsf{x} + \mathsf{y}) - H(\mathsf{x}). \end{aligned}$$

We note that $G(\mathsf{x}) = 0$ for $\mathsf{x} \in X^c$. We, nevertheless, include this term for reasons of consistency. It is clear that $T(\mathsf{x}, 0) = R(\mathsf{x}, 0) = S(\mathsf{x}, 0) = 0$. □

Before proceeding with the proof, we list a result for a linear version of system (6.5) that will be needed in the sequel. It reads as follows.

Lemma 6.5. *Let $T > 0$. Then, the linear problem*

$$\begin{aligned} \partial_t y + \mathsf{P}^s \mathsf{A}_* \mathsf{P}^s y = \mathsf{f}(t) & \quad \textit{in} \quad \Omega, \\ \mathsf{B}_* y = \mathsf{g}(t) & \quad \textit{on} \quad \partial\Omega, \\ y(0) = y_0 & \quad \textit{in} \quad \Omega \end{aligned}$$

admits for each initial value $y_0 = (u_0, F_0, \theta_0, m_0) \in X_{\gamma,\mu}^s$ and each function

$$(\mathsf{f}, \mathsf{g}) \in L_{p,\mu}(J_T; X_0^s) \times \mathbb{F}_\mu(J_T)$$

satisfying the compatibility condition $\mathsf{B}_* y_0 = \mathsf{g}(0)$ a unique solution

$$y \in W_{p,\mu}^1(J_T; X_0^s) \cap L_{p,\mu}(J_T; X_1^s).$$

Moreover, there exists a constant M_0, which is independent of $T \in (0, \infty)$, such that

$$\|y\|_{\mathbb{E}_{1,\mu}(J_T)} \leq M_0 \left(\|\mathsf{f}\|_{\mathbb{E}_{0,\mu}(J_T)} + \|\mathsf{g}\|_{\mathbb{F}_\mu(J_T)} + \|y_0\|_{X_{\gamma,\mu}} \right).$$

Proof. The proof can be reproduced line by line by following the proof of [33, Proposition 3.3].

We will now continue with the proof of Theorem 6.3. Suppose that

$$\mathsf{x}_0 \in B_{X^c}(0,\delta) \quad \text{and} \quad \mathsf{y}_0 \in B_{X^s_{\gamma,\mu}}(0,\delta) \tag{6.6}$$

for a number $\delta > 0$ to be determined later. We already know that (6.4) has a solution $\bar{z}$ with initial value $\bar{z}_0 = \mathsf{x}_0 + \mathsf{y}_0$ on maximal interval of existence $[0, T_+)$, or equivalently, (6.5) has a solution (x, y) on $[0, T_+)$.

As in [32, Theorem 5.3.1], we can show that there exists some constant $C_1 > 0$ such that

$$\begin{aligned} \|T(\mathsf{x},\mathsf{y})\|_{X_0} &\le C_1 \|\mathsf{y}\|_{X_1}, \\ \|R(\mathsf{x},\mathsf{y})\|_{X_0} &\le \Phi(r)\|\mathsf{y}\|_{X_1} \end{aligned} \tag{6.7}$$

for all $\mathsf{x} \in B_{X^c}(0,r)$ and $\mathsf{y} \in B_{X^s_{\gamma,\mu}}(0,r) \cap X_1$ with sufficiently small $r > 0$. We recall here that $\Phi(r) \to 0^+$ as $r \to 0^+$. Define

$$\omega_0 = \frac{1}{2} \inf\{\mathrm{Re}\lambda : \lambda \in \sigma(A_*) \setminus \{0\}\}.$$

For any $\omega \in (0, \omega_0)$, we define the map $e_\omega : L_{1,loc}(\mathbb{R}_+) \to L_{1,loc}(\mathbb{R}_+) : u \mapsto e^{\omega t} u(t)$. For arbitrary $T \in (0, T_+)$, we will establish an estimate of the form

$$\|e_\omega S(\mathsf{x},\mathsf{y})\|_{\mathbb{F}_\mu(J_T)} \le \Phi(r) \|e_\omega \mathsf{y}\|_{\mathbb{E}_{1,\mu}(J_T)}, \tag{6.8}$$

whenever $\|\mathsf{x}(t)\|_{X_0}, \|(\mathsf{x}+\mathsf{y})(t)\|_{X_{\gamma,\mu}} \le r, t \in [0,T]$. Let $\widehat{K}(z) = K(z_* + z) - K(z_*)$. For $z_i \in X_{\gamma,\mu}$, $i = 1, 2$, we have the estimates

$$\begin{aligned} \|\widehat{K}(z_1)\|_{W^{2\mu-2/p}_p(\Omega)} &\le \Phi(\|z_*\|_{X_{\gamma,\mu}} + \|z_1\|_{X_{\gamma,\mu}}) \|z_1\|_{W^{2\mu-2/p}_p(\Omega)}, \\ |(\widehat{K}(z_1) - \widehat{K}(z_2))(x)| &\le \Phi(\|z_*\|_{X_{\gamma,\mu}} + \sum_{i=1,2} \|z_i\|_{X_{\gamma,\mu}}) |(z_1 - z_2)(x)|, \quad x \in \overline{\Omega}, \end{aligned} \tag{6.9}$$

in view of Lemma B.1 and a mean value theorem argument as in Lemma B.2.

We set $\mathsf{x} = (u_1, F_1, \theta_1, m_1)$ and $\mathsf{y} = (u_2, F_2, \theta_2, m_2)$ for $\mathsf{x} \in X^c$ and $\mathsf{y} \in X^s_{\gamma,\mu}$. It holds that

$$S(\mathsf{x},\mathsf{y}) = \nu \cdot \mathrm{tr}_{\partial\Omega}(\widehat{K}(\mathsf{x}+\mathsf{y}) \nabla\theta_2).$$

Note that, in the above computations, we have used the fact that $\nabla\theta_1 = 0$, which follows from the fact that $\mathsf{x} \in X^c = \{0\} \times \{0_3\} \times \mathbb{R} \times \mathbb{R}^3$. We start with the $L_{p,\mu}$-estimate, which reads

$$\begin{aligned} &\|e_\omega S(\mathsf{x},\mathsf{y})\|_{L_{p,\mu}(J_T; L_p(\partial\Omega))} \\ &\le \Phi(r) \|e_\omega \mathrm{tr}_{\partial\Omega}(\nabla\theta_2)\|_{L_{p,\mu}(J_T;L_p(\partial\Omega))} \le \Phi(r)\|e_\omega \mathsf{y}\|_{\mathbb{E}_{1,\mu}(J_T)}. \end{aligned} \tag{6.10}$$

In (6.10), we have used (6.9) and the assumption $\|\mathsf{x}+\mathsf{y}\|_{X_{\gamma,\mu}} \leq r$. $\|e_\omega S(\mathsf{x},\mathsf{y})\|_{L_{p,\mu}(J_T;W_p^{1-1/p}(\partial\Omega))}$ can be estimated in a similar way by observing that $W_p^{1-1/p}(\partial\Omega)$ is a Banach algebra, as $p > 5 > 3$. Let $r = 1/2 - 1/2p$. It follows from Lemma A.4 that

$$[e_\omega S(\mathsf{x},\mathsf{y})]^p_{W^r_{p,\mu}(J_T;L_p(\partial\Omega))}$$
$$\leq C\|e_\omega S(\mathsf{x},\mathsf{y})\|^p_{L_{p,\mu}(J_T;L_p(\partial\Omega))}$$
$$+ C\iint_{B_T^1} \frac{\left\|s^{1-\mu}e^{\omega s}[\widehat{K}(\overline{z}(t)) - \widehat{K}(\overline{z}(s))]\nabla\theta_2(t)\right\|^p_{L_p(\partial\Omega)}}{(t-s)^{1+rp}}\,\mathrm{d}s\,\mathrm{d}t \tag{6.11}$$
$$+ C\iint_{B_T^1} \frac{\left\|s^{1-\mu}e^{\omega s}\widehat{K}(\overline{z}(s))\nabla\left[\theta_2(t) - \theta_2(s)\right]\right\|^p_{L_p(\partial\Omega)}}{(t-s)^{1+rp}}\,\mathrm{d}s\,\mathrm{d}t. \tag{6.12}$$

To estimate (6.11), we recall that $X_0^c \doteq X_1^c$; and observe that for

$$(s,t) \in B_T^1 = \{(s,t) \in (0,T)^2 : 0 < t-s < 1\},$$

we obtain

$$\frac{\|s^{1-\mu}e^{\omega s}(\mathsf{x}(t)-\mathsf{x}(s))\|^p_{L_p(\partial\Omega)}}{(t-s)^{1+\alpha p}} \leq C\frac{\|s^{1-\mu}e^{\omega s}(\mathsf{x}(t)-\mathsf{x}(s))\|^p_{X_0}}{(t-s)^{1+\alpha p}}$$
$$\leq \frac{C}{(t-s)^{1+\alpha p}}\left(\int_s^t \tau^{1-\mu}e^{\omega\tau}\|T(\mathsf{x},\mathsf{y})(\tau)\|_{X_0}\,d\tau\right)^p$$
$$\leq \frac{C}{(t-s)^{1+\alpha p}}\left(\int_s^t \tau^{1-\mu}\|e_\omega \mathsf{y}(\tau)\|_{X_1}\,d\tau\right)^p \tag{6.13}$$
$$\leq C(t-s)^\beta\left(\int_s^{(s+1)\wedge T} \tau^{(1-\mu)p}\|e_\omega \mathsf{y}(\tau)\|^p_{X_1}\,d\tau\right), \tag{6.14}$$

for some constant C which is independent of $T \in [0, T_+)$, where $\beta = (1-\alpha)p-2 > 0$ due to (3.2), and $(s+1)\wedge T := \min\{s+1, T\}$. We have used (6.7) in (6.13) and Hölder's inequality in (6.14). Observe that

$$\iint_{B_T^1} (t-s)^\beta\left(\int_s^{(s+1)\wedge T} \tau^{(1-\mu)p}\|e_\omega \mathsf{y}(\tau)\|^p_{X_1}\,d\tau\right)\mathrm{d}s\,\mathrm{d}t$$
$$= \int_0^T\int_s^{(s+1)\wedge T}(t-s)^\beta\left(\int_s^{(s+1)\wedge T} \tau^{(1-\mu)p}\|e_\omega \mathsf{y}(\tau)\|^p_{X_1}\,d\tau\right)\mathrm{d}t\,\mathrm{d}s$$
$$\leq \left(\int_0^T\int_s^{(s+1)\wedge T} \tau^{(1-\mu)p}\|e_\omega \mathsf{y}(\tau)\|^p_{X_1}\,d\tau\,\mathrm{d}s\right)\left(\int_0^1 t^\beta\,\mathrm{d}t\right)$$
$$= \left(\int_0^T \tau^{(1-\mu)p}\|e_\omega \mathsf{y}(\tau)\|^p_{X_1}\,d\tau\int_{(\tau-1)\vee 0}^\tau \mathrm{d}s\right)\left(\int_0^1 t^\beta\,\mathrm{d}t\right)$$

$$\leq \frac{1}{\beta+1}\|e_\omega \mathsf{y}\|^p_{\mathbb{E}_{1,\mu}(J_T)}, \tag{6.15}$$

where $(\tau-1)\vee 0 = \max\{\tau-1,0\}$. Employing (6.9), (6.14), (6.15) and Lemma A.4, we have

$$\begin{aligned}
&\iint_{B_T^1} \frac{\left\|s^{1-\mu}e^{\omega s}[\widehat{K}(\overline{z}(t)) - \widehat{K}(\overline{z}(s))]\nabla\theta_2(t)\right\|^p_{L_p(\partial\Omega)}}{(t-s)^{1+rp}}\,\mathrm{d}s\,\mathrm{d}t \\
&\leq \Phi(r)\iint_{B_T^1} \frac{\left\|s^{1-\mu}e^{\omega s}(\mathsf{x}(t)-\mathsf{x}(s))\right\|^p_{L_p(\partial\Omega)}}{(t-s)^{1+rp}}\,\mathrm{d}s\,\mathrm{d}t + \Phi(r) \\
&\iint_{B_T^1} \frac{\left\|s^{1-\mu}e^{\omega s}(\mathsf{y}(t)-\mathsf{y}(s))\right\|^p_{L_p(\partial\Omega)}}{(t-s)^{1+rp}}\,\mathrm{d}s\,\mathrm{d}t \\
&\leq \Phi(r)\|e_\omega \mathsf{y}\|^p_{\mathbb{E}_{1,\mu}(J_T)},
\end{aligned}$$

where we have used the fact that $\|\nabla\theta_2(t)\|_\infty \leq C\|\mathsf{y}(t)\|_{X_{\gamma,\mu}} \leq Cr$. The estimate for (6.12) is a direct consequence of (6.9):

$$\begin{aligned}
&\iint_{B_T^1} \frac{\left\|s^{1-\mu}e^{\omega s}\widehat{K}(\overline{z}(s))\nabla\left[\theta_2(t)-\theta_2(s)\right]\right\|^p_{L_p(\partial\Omega)}}{(t-s)^{1+rp}}\,\mathrm{d}s\,\mathrm{d}t \\
&\quad \leq \Phi(r)\iint_{B_T^1} \frac{\left\|s^{1-\mu}e^{\omega s}\nabla\left[\theta_2(t)-\theta_2(s)\right]\right\|^p_{L_p(\partial\Omega)}}{(t-s)^{1+rp}}\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

Summarizing the above discussion and applying (6.10) and Lemma A.4, we have

$$\begin{aligned}
[e_\omega S(\mathsf{x},\mathsf{y})]^p_{W^r_{p,\mu}(J_T;L_p(\partial\Omega))} &\leq \Phi(r)\|e_\omega \mathsf{y}\|^p_{\mathbb{E}_{1,\mu}(J_T)} + \Phi(r)\|e_\omega\nabla\theta_2\|^p_{W^r_{p,\mu}(J_T;L_p(\partial\Omega))} \\
&\leq \Phi(r)\|e_\omega \mathsf{y}\|^p_{\mathbb{E}_{1,\mu}(J_T)}.
\end{aligned}$$

In the last step, we have used the embedding

$$W^{1-1/2p}_{p,\mu}(J_T;L_p(\partial\Omega)) \cap L_{p,\mu}(J_T;W^{2-1/p}_p(\partial\Omega)) \hookrightarrow W^r_{p,\mu}(J_T;W^1_p(\partial\Omega)),$$

see for instance [28, Proposition 3.2], and [28, Theorem 4.5]. This yields (6.8).

Fix $r>0$ so that estimates (6.7) and (6.8) hold. We put

$$t_0 = \sup\{t\in(0,T_+):\ \|\mathsf{x}(\tau)\|_{X_0}, \|\mathsf{y}(\tau)\|_{X_{\gamma,\mu}} \leq r,\ \tau\in[0,t]\}.$$

Assume that $t_0 < T_+$. Then, Lemma 6.5 implies

$$\begin{aligned}
\|e_\omega \mathsf{y}\|_{\mathbb{E}_{1,\mu}(J_{t_0})} &\leq M_0\left(\|\mathsf{y}_0\|_{X_{\gamma,\mu}} + \|e_\omega S(\mathsf{x},\mathsf{y})\|_{\mathbb{F}_\mu(J_{t_0})} + \|e_\omega R(\mathsf{x},\mathsf{y})\|_{\mathbb{E}_{0,\mu}(J_{t_0})}\right) \\
&\leq M_0\|\mathsf{y}_0\|_{X_{\gamma,\mu}} + \Phi(r)\|e_\omega \mathsf{y}\|_{\mathbb{E}_{1,\mu}(J_{t_0})}.
\end{aligned}$$

Choosing $r>0$ sufficiently small so that $\Phi(r)<1/2$ yields

$$\|e_\omega \mathsf{y}\|_{\mathbb{E}_{1,\mu}(J_{t_0})} \leq 2M_0\|\mathsf{y}_0\|_{X_{\gamma,\mu}},$$

which further implies that

$$\|e_\omega \mathsf{y}\|_{C([0,t_0];X_{\gamma,\mu})} \le M_1 \|\mathsf{y}_0\|_{X_{\gamma,\mu}}.$$

We can derive an estimate for x by using (6.5) and (6.7):

$$\begin{aligned}\|\mathsf{x}(t)\|_{X_0} &\le \|\mathsf{x}_0\|_{X_0} + \int_0^t \|T(\mathsf{x},\mathsf{y})(\tau)\|_{X_0}\, d\tau \\ &\le \|\mathsf{x}_0\|_{X_0} + C_1 \int_0^t e^{-\omega\tau}\tau^{\mu-1}\|\tau^{1-\mu}e^{\omega\tau}\mathsf{y}(\tau)\|_{X_1}\, d\tau \\ &\le \|\mathsf{x}_0\|_{X_0} + C\|e_\omega \mathsf{y}\|_{\mathbb{E}_{1,\mu}(J_{t_0})} \le \|\mathsf{x}_0\|_{X_0} + M_2\|\mathsf{y}_0\|_{X_{\gamma,\mu}}.\end{aligned}$$

In the last line we employed Hölder's inequality and $\mu > 1/p$. By choosing $\delta < r/2(1 + M_1 + M_2)$, where δ was introduced in (6.6), we have for all $\mathsf{x}_0 \in B_{X^c}(0,\delta)$ and $\mathsf{y}_0 \in B_{X^s_{\gamma,\mu}}(0,\delta)$ and all $t \in [0, t_0)$

$$\|\mathsf{x}(t)\|_{X_0} + \|\mathsf{y}(t)\|_{X_{\gamma,\mu}} \le \|\mathsf{x}_0\|_{X_0} + (M_1 + M_2)\|\mathsf{y}_0\|_{X_{\gamma,\mu}} \le r/2,$$

a contradiction to the definition of t_0. Therefore, $t_0 = T_+$. With the choice $\delta < r/2(1 + M_1 + M_2)$, the above discussion shows that there exists a constant $M_3 > 0$ such that for any $t_1 \in (0, T_+)$,

$$\|\overline{z}\|_{C([0,t_1];X_{\gamma,\mu})} + \|\overline{z}\|_{\mathbb{E}_{1,\mu}(J_{t_1})} \le M_3.$$

Let $\tau \in (0, t_1)$ be fixed and let t be any number in $[\tau, t_1]$. Then, we have

$$\begin{aligned}\|\overline{z}(t)\|_{X_{\gamma,1}} &\le \sup_{s\in[\tau,t_1]} \|\overline{z}(s)\|_{X_{\gamma,1}} \le C(\tau)\|\overline{z}\|_{\mathbb{E}_{1,1}([\tau,t_1])} \le C(\tau)\tau^{\mu-1}\|\overline{z}\|_{\mathbb{E}_{1,\mu}([\tau,t_1])} \\ &\le C(\tau)\tau^{\mu-1}M_3.\end{aligned}$$

This implies that $\overline{z} \in BC([\tau, T_+), X_{\gamma,1})$. Theorem 6.1 then implies that $T_+ = \infty$. The rest of the proof is exactly the same as part (f) of the proof of [32, Theorem 5.3.1]. □

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Declarations

Conflict of interest The authors assert that there is no conflict of interest to declare.

Appendix A: Properties of fractional Sobolev spaces with temporal weights

For $r \in (0, 1)$, fractional Sobolev spaces with temporal weight can also be defined by means of interpolation. It then holds that

$$W^r_{p,\mu}(J_T; X) \doteq \Big(L_{p,\mu}(J_T; X), W^1_{p,\mu}(J_T; X)\Big)_{r,p},$$

where the symbol $\doteq$ means equivalent norms; see [26, Proposition 1.1.13], or [28, equation (2.6)]. The corresponding norm is called the interpolation norm of $W^r_{p,\mu}(J_T; X)$. It is pointed out in [26, Remark 1.1.15] that the equivalence constant between the intrinsic norm (1.5) and the interpolation norm of $W^r_{p,\mu}(J_T; X)$ blows up as $T \to 0^+$. Using interpolation norms can cause difficulties in obtaining uniform estimates for nonlinear terms on short time intervals $(0, T)$ that are independent of T. This difficulty can often be circumvented by using intrinsic norms.

In this section, we establish some useful results for fractional Sobolev spaces with temporal weights by exclusively using intrinsic norms. These results are also interesting in their own right. Analogous results have been obtained in [26], see also [28], by using interpolation norms. For instance, it is shown in [26, Lemma 1.1.15] that there exists an extension operator $\mathcal{E}_T : {}_0W^r_{p,\mu}(J_T; X) \to {}_0W^r_{p,\mu}(\mathbb{R}_+; X)$ whose norm is independent of T, where both spaces are equipped with the corresponding interpolation norms. The merit of Proposition A.4 lies in the fact that we can completely rely on intrinsic norms. This greatly facilitates deriving estimates for nonlinear boundary terms.

The results obtained in this section employed in obtaining estimates for nonlinear mappings, but are also of independent interest. We recall that

$${}_0W^r_{p,\overline{\mu}}(J_T; X) := \{u \in W^r_{p,\overline{\mu}}(J_T; X) : \gamma_0 u = 0\},$$

where $r, \overline{\mu} \in (1/p, 1]$ with $r + \overline{\mu} > 1 + 1/p$, and X is a Banach space; see [28, Proposition 2.10].

Lemma A.1. *Let X be a Banach space. Given $p \in (1,\infty)$ and $r,\mu \in (1/p,1]$ such that $r+\mu > 1+1/p$, there exists a constant $C>0$, which is independent of $T \in (0,\infty]$, such that*

$$\left(\int_0^T t^{(1-\mu-r)p}\|u(t)\|_X^p\,\mathrm{d}t\right)^{1/p} \le C\|u\|_{W^r_{p,\mu}(J_T;X)}$$

for all $u \in {}_0W^r_{p,\mu}(J_T;X)$, where ${}_0W^r_{p,\mu}(J_T;X)$ is equipped with the intrinsic norm.

Proof. The case $r \in \{0,1\}$ follows from the definition of $L_{p,\mu}(J_T;X)$ and [26, Lemma 1.1.2(b)]. When $r \in (0,1)$,

$$\begin{aligned}&\left(\int_0^T t^{(1-\mu-r)p}\|u(t)\|_X^p\,\mathrm{d}t\right)^{1/p}\\&=\left(\int_0^T\left(t^{-\mu-r}\int_0^t\|u(t)\|_X\,\mathrm{d}s\right)^p\mathrm{d}t\right)^{1/p}\\&\le\left(\int_0^T\left(t^{-\mu-r}\int_0^t\|u(t)-u(s)\|_X\,\mathrm{d}s\right)^p\mathrm{d}t\right)^{1/p}\\&\quad+\left(\int_0^T\left(t^{-\mu-r}\int_0^t\|u(s)\|_X\,\mathrm{d}s\right)^p\mathrm{d}t\right)^{1/p}.\end{aligned}\tag{A.1}$$

We will use Hölder's inequality to estimate the first term in (A.1) as follows:

$$\begin{aligned}&\left(\int_0^T\left(t^{-\mu-r}\int_0^t\|u(t)-u(s)\|_X\,\mathrm{d}s\right)^p\mathrm{d}t\right)^{1/p}\\&\le\left(\int_0^T t^{(-\mu-r)p}\left(\int_0^t s^{(1-\mu)p}\|u(t)-u(s)\|_X^p\,\mathrm{d}s\right)\left(\int_0^t s^{(\mu-1)p'}\,\mathrm{d}s\right)^{p/p'}\mathrm{d}t\right)^{1/p}\\&\le C\left(\int_0^T t^{-1-rp}\left(\int_0^t s^{(1-\mu)p}\|u(t)-u(s)\|_X^p\,\mathrm{d}s\right)\mathrm{d}t\right)^{1/p}\\&\le C[u]_{W^r_{p,\mu}(J_T;X)}.\end{aligned}$$

In the last line, we used that $1/t < 1/(t-s)$ for $s \in (0,t)$. Observe that it follows from the condition $\mu \in (1/p,1]$ that $(\mu-1)p' > -1$. To estimate the second term in (A.1), we will apply Hardy's inequality, c.f. [32, Lemma 3.4.5], to obtain

$$\left(\int_0^T\left(t^{-\mu-r}\int_0^t\|u(s)\|_X\,\mathrm{d}s\right)^p\mathrm{d}t\right)^{1/p} \le \frac{1}{(\mu+r-1/p)}\left(\int_0^T t^{(1-\mu-r)p}\|u(t)\|_X^p\,\mathrm{d}t\right)^{1/p}.$$

Hence, we have shown that

$$\begin{aligned}\left(\int_0^T t^{(1-\mu-r)p}\|u(t)\|_X^p\,\mathrm{d}t\right)^{1/p} &\le C\|u\|_{W^r_{p,\mu}(J_T;X)}+\frac{1}{(\mu+r-1/p)}\\&\quad\left(\int_0^T t^{(1-\mu-r)p}\|u(t)\|_X^p\,\mathrm{d}t\right)^{1/p}.\end{aligned}$$

In view of the condition $r+\mu > 1+1/p$, the asserted estimate then follows. □

Lemma A.2. *Suppose* $\mu \in [0, 1]$. *Then, we have*

$$\left(t^{1-\mu} - s^{1-\mu}\right)^p \leq t^{-\mu p}(t-s)^p \quad \textit{and} \quad |t^{\mu-1} - s^{\mu-1}|^p \leq s^{(\mu-1)p} t^{-p}(t-s)^p, \quad 0 < s < t < \infty.$$

Proof. The assertions are clear for $\mu \in \{0, 1\}$. In case $\mu \in (0, 1)$, we obtain

$$(t^{1-\mu} - s^{1-\mu})^p = t^{(1-\mu)p}\left(1 - (s/t)^{1-\mu}\right)^p \leq t^{(1-\mu)p}\left(1 - (s/t)\right)^p = t^{-\mu p}(t-s)^p.$$

This estimate, in turn, yields

$$|t^{\mu-1} - s^{\mu-1}|^p = s^{(\mu-1)p} t^{(\mu-1)p}(t^{1-\mu} - s^{1-\mu})^p \leq s^{(\mu-1)p} t^{-p}(t-s)^p.$$

□

For $u \in L_{1,\mathrm{loc}}(J_T; X)$, we define $(\Phi_\mu u)(t) := t^{1-\mu} u(t)$, see [31]. It is then clear that

$$\Phi_\mu : L_{p,\mu}(J_T; X) \to L_p(J_T; X) \quad \text{is an isometric isomorphism,} \tag{A.2}$$

and its inverse Φ_μ^{-1} is given by $(\Phi_\mu^{-1} v)(t) = t^{\mu-1} v(t)$. The next result shows that Φ_μ induces an isomorphism for the Sobolev spaces ${}_0W^r_{\mu,p}(J_T; X)$.

Lemma A.3. *Let* X *be a Banach space. Suppose that* $r, \mu \in (1/p, 1]$ *and* $r + \mu > 1 + 1/p$. *Then, it holds that*

$$\Phi_\mu \in \mathcal{L}is({}_0W^r_{p,\mu}(J_T; X), {}_0W^r_p(J_T; X)).$$

Moreover, there exists a constant C *which is independent of* $T \in (0, \infty]$ *such that*

$$\|\Phi_\mu u\|_{W^r_{p,\mu}(J_T;X)} \leq C\|u\|_{W^r_p(J_T;X)}, \qquad \|\Phi_\mu^{-1} v\|_{W^r_p(J_T;X)} \leq C\|v\|_{W^r_{p,\mu}(J_T;X)}, \tag{A.3}$$

where the spaces are equipped with their respective intrinsic norms.

Proof. The first part of the assertion has been established in [28, Lemma 2.3], where the spaces are equipped with their respective interpolation norms.

We will now establish the uniform estimates in (A.3) for intrinsic norms. The case $r = 1$ follows readily from Lemma A.1. For the reader's convenience, we include a proof (see also [26, Lemma 1.1.3]). Suppose $u \in W^1_{p,\mu}(J_T; X)$. Then, we obtain

$$\begin{aligned}\|(\Phi_\mu u)'\|_{L_p(J_T;X)} &\leq \left(\int_0^T \|t^{1-\mu} u'(t)\|_X^p \,\mathrm{d}t\right)^{1/p} + (1-\mu)\left(\int_0^T t^{-\mu p}\|u(t)\|_X^p \,\mathrm{d}t\right)^{1/p} \\ &\leq C\|u\|_{W^1_{p,\mu}(J_T;X)},\end{aligned}$$

where we used Lemma A.1 with $r = 1$. Suppose now that $v \in {}_0W^1_{p,\mu}(J_T; X)$. Then, we obtain

$$\|(\Phi_\mu^{-1} v)'\|_{L_{p,\mu}(J_T;X)} \leq \left(\int_0^T \|v'(t)\|_X^p \,\mathrm{d}t\right)^{1/p} + |\mu - 1|\left(\int_0^T t^{-1}\|v\|_X^p \,\mathrm{d}t\right)^{1/p}$$

$$\leq \|v\|_{W^1_p(J_T;X)},$$

where we employed, once more, Lemma A.1 with $r = \mu = 1$. These estimates together with (A.2) imply the assertion.

We will now consider the case $r < 1$ and $r + \mu > 1 + 1/p$. Suppose $u \in {}_0W^r_{p,\mu}(J_T; X)$. Then, we obtain

$$\begin{aligned}
[\Phi_\mu u]_{W^r_p(J_T;X)} &= \left(\int_0^T \int_0^t \frac{\|(\Phi_\mu u)(t) - (\Phi_\mu u)(s)\|_X^p}{(t-s)^{1+rp}} \,\mathrm{d}s\, dt\right)^{1/p} \\
&\leq \left(\int_0^T \int_0^t s^{p(1-\mu)} \frac{\|u(t) - u(s)\|_X^p}{(t-s)^{1+rp}} \,\mathrm{d}s\, dt\right)^{1/p} \\
&\quad + \left(\int_0^T \int_0^t \frac{(t^{1-\mu} - s^{1-\mu})^p}{(t-s)^{1+rp}} \|u(t)\|_X^p \,\mathrm{d}s\, dt\right)^{1/p} \\
&\leq [u]_{W^r_{p,\mu}(J_T;X)} + \left(\int_0^T \int_0^t t^{-\mu p}(t-s)^{(1-r)p-1} \|u(t)\|_X^p \,\mathrm{d}s\, dt\right)^{1/p} \\
&\leq [u]_{W^r_{p,\mu}(J_T;X)} + c(r,p)\left(\int_0^T t^{(1-\mu-r)p} \|u(t)\|_X^p \, dt\right)^{1/p} \qquad \text{(A.4)} \\
&\leq C\|u\|_{W^r_{p,\mu}(J_T;X)}. \qquad \text{(A.5)}
\end{aligned}$$

We used Lemma A.2 in (A.4) and Lemma A.1 in (A.5).
Suppose that $v \in {}_0W^r_p(J_T; X)$. Then, we obtain

$$\begin{aligned}
[(\Phi_\mu)^{-1} v]_{W^r_{p,\mu}(J_T;X)} &= \left(\int_0^T \int_0^t s^{(1-\mu)p} \frac{\|(\Phi_\mu^{-1} v)(t) - (\Phi_\mu^{-1} v)(s)\|_X^p}{(t-s)^{1+rp}} \,\mathrm{d}s\, dt\right)^{1/p} \\
&\leq \left(\int_0^T \int_0^t \frac{\|v(t) - v(s)\|_X^p}{(t-s)^{1+rp}} \,\mathrm{d}s\, dt\right)^{1/p} \\
&\quad + \left(\int_0^T \int_0^t s^{(1-\mu)p} \frac{|t^{\mu-1} - s^{\mu-1}|^p}{(t-s)^{1+rp}} \|v(t)\|_X^p \,\mathrm{d}s\, dt\right)^{1/p} \\
&\leq [u]_{W^r_p(J_T;X)} + c(r,p)\left(\int_0^T t^{-rp} \|u(t)\|_X^p \, dt\right)^{1/p} \qquad \text{(A.6)} \\
&\leq C\|u\|_{W^r_p(J_T;X)}. \qquad \text{(A.7)}
\end{aligned}$$

Here we used, once more, Lemma A.2 in (A.6) and Lemma A.1 in (A.7). □

Proposition A.4. *Let X be a Banach space. Suppose $r, \mu \in (1/p, 1]$ and $r + \mu > 1 + 1/p$. Then, there exists an extension operator:*

$$\mathcal{E}_{J_T} : {}_0W^r_{p,\mu}(J_T; X) \to {}_0W^r_{p,\mu}(\mathbb{R}_+; X)$$

such that its norm is independent of $T \in (0, \infty]$, where the spaces are equipped with their intrinsic norms.

Proof. We define the extension operator by

$$\mathcal{E}_{J_T} u(t) := \begin{cases} u(t) & \text{for } 0 < t \le T \\ \left(\frac{2T-t}{t}\right)^{1-\mu} u(2T-t) & \text{for } T < t \le 2T \\ 0 & \text{for } 2T < t. \end{cases}$$

The statement follows from Lemma A.3, [30, Proposition 6.1], and the commutativity of the diagram

$$\begin{array}{ccc} {}_0W^r_{p,\mu}(J_T; X) & \xrightarrow{\Phi_\mu} & {}_0W^r_p(J_T; X) \\ \downarrow \mathcal{E}_{J_T} & & \downarrow \mathcal{E}_T \\ {}_0W^r_{p,\mu}(\mathbb{R}_+; X) & \xleftarrow{\Phi_\mu^{-1}} & {}_0W^r_p(\mathbb{R}_+; X), \end{array}$$

where the extension operator $\mathcal{E}_T$ on the right side is defined in [30, Proposition 6.1]. □

The following result is used in Sect. 6 in order to show stability of (constant) equilibria.

Lemma A.4. *Let $T > 0$, $r \in (0, 1)$, $\omega \in \mathbb{R}$, and $\mu \in (1/p, 1]$. We then set*

$$B_T = \{(s,t) \in (0,T)^2 : 0 < s < t\} \quad \text{and} \quad B_T^1 = \{(s,t) \in (0,T)^2 : 0 < t - s < 1\}.$$

Suppose that X is a Banach space and $u \in W^r_{p,\mu}(J_T; X)$. Then,

$$\begin{aligned}[][e_\omega u]_{W^r_{p,\mu}(J_T;X)} &\le C\|e_\omega u\|_{L_{p,\mu}(J_T;X)} + \left(\iint_{B_T^1} \frac{\|s^{1-\mu} e^{\omega s}(u(t) - u(s))\|_X^p}{(t-s)^{1+rp}}\, \mathrm{d}s\, \mathrm{d}t\right)^{1/p} \\ &\le C\|e_\omega u\|_{W^r_{p,\mu}(J_T;X)},\end{aligned}$$

where the constant $C = C(p, r, \omega)$ is independent of T and $e_\omega : L_{1,loc}(\mathbb{R}_+) \to L_{1,loc}(\mathbb{R}_+) : u \mapsto e^{\omega t} u$.

Proof. Using (1.5), we estimate as in [22, Lemma 11] and obtain

$$\begin{aligned}&[e_\omega u]_{W^r_{p,\mu}(J_T;X)} \\ &\le \left(\iint_{B_T \setminus B_T^1} s^{p(1-\mu)} \frac{\|e^{\omega t} u(t) - e^{\omega s} u(s)\|_X^p}{(t-s)^{1+rp}}\, \mathrm{d}s\, dt\right)^{1/p} \\ &\quad + \left(\iint_{B_T^1} s^{p(1-\mu)} \frac{\|e^{\omega t} u(t) - e^{\omega s} u(s)\|_X^p}{(t-s)^{1+rp}}\, \mathrm{d}s\, dt\right)^{1/p} \\ &\le \left(\int_0^T \int_0^{t-1} \frac{\|t^{1-\mu}\, e^{\omega t} u(t)\|_X^p}{(t-s)^{1+rp}}\, \mathrm{d}s\, dt\right)^{1/p} + \left(\int_0^T \int_{s+1}^T \frac{\|s^{1-\mu}\, e^{\omega s} u(s)\|_X^p}{(t-s)^{1+rp}}\, \mathrm{d}t\, ds\right)^{1/p} \\ &\quad + \left(\iint_{B_T^1} s^{p(1-\mu)} e^{\omega t p} \|u(t)\|_X^p \frac{|e^{-\omega(t-s)} - 1|^p}{(t-s)^{1+rp}}\, \mathrm{d}s\, dt\right)^{1/p}\end{aligned}$$

$$
\begin{aligned}
&+\left(\iint_{B_T^1} s^{p(1-\mu)} e^{\omega s p} \frac{\|u(t)-u(s)\|_X^p}{(t-s)^{1+rp}}\,\mathrm{d}s\,dt\right)^{1/p}\\
&\le \|e_\omega u\|_{L_{p,\mu}(J_T;X)}\left[2\left(\int_1^\infty \frac{d\tau}{\tau^{1+rp}}\right)^{1/p}+c(\omega)\left(\int_0^1 \frac{d\tau}{\tau^{1+(r-1)p}}\right)^{1/p}\right]\\
&+\left(\iint_{B_T^1} s^{p(1-\mu)} e^{\omega s p} \frac{\|u(t)-u(s)\|_X^p}{(t-s)^{1+rp}}\,\mathrm{d}s\,dt\right)^{1/p}\\
&\le C\|e_\omega u\|_{L_{p,\mu}(J_T;X)}+\left(\iint_{B_T^1} s^{p(1-\mu)} e^{\omega s p} \frac{\|u(t)-u(s)\|_X^p}{(t-s)^{1+rp}}\,\mathrm{d}s\,dt\right)^{1/p}.
\end{aligned}
$$

In the derivation above, we have used that $s < t$ for $(s,t) \in B_T$. □

Our next result deals with multiplication properties in weighted Sobolev spaces.

Lemma A.5. *Let $T_* > 0$ be given.*

(i) *There exists a constant $C > 0$, which is independent of $T \in (0, T_*]$, such that*

$$\|uv\|_{\mathbb{F}_\mu(J_T)} \le C\|u\|_{\mathbb{F}_\mu(J_T)}\|v\|_{\mathbb{F}_\mu(J_T)}, \quad \textit{for all } u,v \in {}_0\mathbb{F}_\mu(J_T).$$

(ii) *There exists a constant $C > 0$, which is independent of $T \in (0, T_*]$, such that*

$$\|uv\|_{\mathbb{F}_\mu(J_T)} \le C\|u\|_{\mathbb{F}_\mu(J_T)}\|v\|_{\mathbb{F}_{1,\mu}(J_T)}, \quad \textit{for all } (u,v) \in \mathbb{F}_\mu(J_T)\times {}_0\mathbb{F}_{1,\mu}(J_T),$$

where $\mathbb{F}_{1,\mu}(J_T)$ is defined as

$$\mathbb{F}_{1,\mu}(J_T) := W_{p,\mu}^{1-1/2p}(J_T; L_p(\partial\Omega)) \cap C([0,T]; W_p^{2\mu-3/p}(\partial\Omega)). \tag{A.8}$$

Proof. (i) The assertion follows from the fact that $\mathbb{F}_\mu(J_T)$ is a Banach algebra and Proposition A.4. See also [26, Lemma 1.3.23].
(ii) To explain the occurrence of the space $C([0,T]; W_p^{2\mu-3/p}(\partial\Omega))$ in (A.8), we note that

$$W_{p,\mu}^{1-1/2p}(J_T; L_p(\partial\Omega)) \cap L_{p,\mu}(J_T; W_p^{2-1/p}(\partial\Omega)) \hookrightarrow C([0,T]; W_p^{2\mu-3/p}(\partial\Omega))$$

see [28, equation (4.10)]. It is an easy task to check that

$$\|uv\|_{L_{p,\mu}(J_T;W_p^{1-1/p}(\partial\Omega))} \le C\|v\|_{L_\infty(J_T;W_p^{1-1/p}(\partial\Omega))}\|u\|_{L_{p,\mu}(J_T;W_p^{1-1/p}(\partial\Omega))}$$

for some $C > 0$ independent of $T \in (0, T_*]$. In addition, one has

$$
\begin{aligned}
&\|uv\|_{W_{p,\mu}^{1/2-1/2p}(J_T;L_p(\partial\Omega))}\\
&\le \|v\|_{C([0,T]\times\overline{\Omega})}\|u\|_{L_{p,\mu}(J_T;L_p(\partial\Omega))}+[uv]_{W_{p,\mu}^{1/2-1/2p}(J_T;L_p(\partial\Omega))}.
\end{aligned}\tag{A.9}
$$

We can derive from Proposition A.4 and equation (1.4) in [29, Theorem 1.1] (by choosing $p_i = q_i = p$ for $i = 0, 1$, $\gamma_0 = (1-\mu)p$, and $\gamma_1 = 0$) that

$${}_0W_{p,\mu}^{1-1/2p}(J_T; L_p(\partial\Omega)) \hookrightarrow {}_0W_p^{s_1}(J_T; L_p(\partial\Omega))$$

for $s_1 = \mu - 1/p$ with embedding constant independent of $T \in (0, T_*]$. Due to assumption (3.2), we have $s_1 > 1/2 + 2/p$. This implies

$$ {}_0\mathbb{F}_{1,\mu}(J_T) \hookrightarrow C^\sigma([0,T]; L_p(\partial\Omega)) $$

for some $\sigma > 1/2 + 1/p$ with embedding constant independent of T. Therefore, the second term on the right-hand side of (A.9) can be estimated as follows:

$$
\begin{aligned}
&[uv]^p_{W^{1/2-1/2p}_{p,\mu}(J_T;L_p(\partial\Omega))}\\
&\quad\le \int_0^T\int_0^t s^{p(1-\mu)} \frac{\|u(t)v(t)-u(s)v(s)\|^p_{L_p(\partial\Omega)}}{(t-s)^{1/2+p/2}}\,\mathrm{d}s\mathrm{d}t\\
&\quad\le \|v\|^p_{C([0,T]\times\overline{\Omega})}[u]^p_{W^{1/2-1/2p}_{p,\mu}(J_T;L_p(\partial\Omega))}\\
&\qquad+\int_0^T\int_0^t s^{p(1-\mu)}\|u(s)\|^p_{C(\overline{\Omega})}\frac{\|v(t)-v(s)\|^p_{L_p(\partial\Omega)}}{(t-s)^{1/2+p/2}}\,\mathrm{d}s\mathrm{d}t\\
&\quad\le \|v\|^p_{C([0,T]\times\overline{\Omega})}[u]^p_{W^{1/2-1/2p}_{p,\mu}(J_T;L_p(\partial\Omega))}\\
&\qquad+\|v\|^p_{C^\sigma([0,T];L_p(\partial\Omega))}\int_0^T\int_0^t s^{p(1-\mu)}\|u(s)\|^p_{C(\overline{\Omega})}(t-s)^{\sigma p-(1/2+p/2)}\,\mathrm{d}s\mathrm{d}t\\
&\quad\le \|v\|^p_{C([0,T]\times\overline{\Omega})}[u]^p_{W^{1/2-1/2p}_{p,\mu}(J_T;L_p(\partial\Omega))}+C_1\|v\|^p_{C^\sigma([0,T];L_p(\partial\Omega))}\|u\|^p_{L_{p,\mu}(J_T;C(\overline{\Omega}))}
\end{aligned}
$$

for some constant $C_1 = C_1(T) > 0$ that is uniform in $T \in (0, T_*]$. This implies

$$ [uv]_{W^{1/2-1/2p}_{p,\mu}(J_T;L_p(\partial\Omega))} \le C\|v\|_{{}_0\mathbb{F}_{1,\mu}(J_T)}\|v\|_{\mathbb{F}_\mu(J_T)}, $$

where $C = C(T)$ is uniform in $T \in (0, T_*]$. □

Appendix B: Properties of nonlinear maps

In this section, we establish some mapping properties for the nonlinear operators in (1.1). Our first step is to study the Nemyskii operators induced by the functions in (2.1).

Lemma B.1. *Suppose* $\varphi \in C^5(\mathbb{R})$ *and* $X \in \{W^{2\mu-2/p}_p(\Omega), W^{2\mu+1-2/p}_p(\Omega), \mathbb{E}^k_{2,\mu}(J_T), \mathbb{F}_\mu(J_T)\}$*, where*

$$ \mathbb{E}^k_{2,\mu}(J_T) := W^1_{p,\mu}(J_T; W^k_p(\Omega)) \cap L_{p,\mu}(J_T; W^{k+2}_p(\Omega)), \quad k = 0, 1. $$

Then, the Nemyskii operator induced by φ*, still denoted by* φ*, satisfies*

$$ \varphi \in C^1(X). $$

Moreover, given $T_* > 0$

$$ \|\varphi(u)\|_{\mathbb{F}_\mu(J_T)} \le C\left(\|\varphi'(u)\|_\infty\|u\|_{\mathbb{F}_\mu(J_T)} + \|\varphi(u)\|_\infty\right), \quad u \in \mathbb{F}_\mu(J_T). \qquad \text{(B.1)} $$

The constant $C > 0$ *is uniform with respect to* $T \in (0, T_*]$.

Proof. The mapping property $\varphi \in C^1(\mathbb{E}^k_{2,\mu}(J_T))$ can be proved via direct computations and the fact that $\varphi \in C^5(\mathbb{R})$. It follows from Lemma 3.1(b) that there exists a bounded right inverse γ_0^c for the initial trace operator

$$\gamma_0 : \mathbb{E}^0_{2,\mu}(J_T) \to W_p^{2\mu-2/p}(\Omega).$$

The C^1-continuity of φ in $W_p^{2\mu-2/p}(\Omega)$ then follows from the relationship

$$\varphi(u) = \gamma_0\varphi(\gamma_0^c(u)), \quad u \in W_p^{2\mu-2/p}(\Omega).$$

The case $X = W_p^{2\mu+1-2/p}(\Omega)$ follows from a similar argument. The assertion in (B.1) has been proved in [26, Lemma 4.2.3(a)]. A close look at its proof shows that the constant in [26, Lemma 4.2.3(a)] is uniform with respect to $T \in (0, T_*]$. The C^1-continuity of φ in $\mathbb{F}_\mu(J_T)$ can be derived from (B.1) by a mean value theorem argument and the fact that $\mathbb{F}_\mu(J_T)$ is a Banach algebra. □

Next, we will establish some relevant mapping properties of the operators in (3.7). For the analysis below, note that by Proposition A.4 and [28, Theorems 4.2 and 4.5], there exists a constant $C > 0$ such that

$$\|\mathrm{tr}_{\partial\Omega} v\|_{\mathbb{F}_{1,\mu}(J_T)} \le C\|v\|_{\mathbb{E}^0_{2,\mu}(J_T)}, \quad v \in \mathbb{E}^0_{2,\mu}(J_T), \tag{B.2}$$

where the embedding constant is independent of T if $v \in {}_0\mathbb{E}^0_{2,\mu}(J_T)$.

Suppose that $\phi_1 \in C^5(\mathbb{R}^{16})$, $\phi_2 \in C^5(\mathbb{R}^{48})$ and $\phi_3 \in C^5(\mathbb{R}^9)$. In order to derive an estimate for

$$\|\mathcal{A}(z_1 + z_2) - \mathcal{A}(z_1) - \mathcal{A}'(z_1)z_2\|_{\mathbb{E}_{0,\mu}(J_T)}$$

for proper functions $z_1, z_2 \in \mathbb{E}_{1,\mu}(J_T)$, we will consider five types of mappings, given by

$$\begin{aligned}
G_1(z) &= \phi_1(z)\phi_2(\partial z),\\
G_2(z) &= \phi_1(z)\partial_{ij} z,\\
G_3(z) &= \phi_1(z)\phi_3(\partial m),\\
G_4(z) &= \phi_1(z)\phi_3(\partial m)\partial_{ij} m,\\
G_5(z) &= \phi_1(z)|\Delta m|^2,
\end{aligned} \tag{B.3}$$

where for any function $z = (u, F, \theta, m) \in C^1(\Omega, \mathbb{R}^{16})$, we define $\partial z \in C(\Omega, \mathbb{R}^{48})$ by $\partial z = (\partial_1 z, \partial_2 z, \partial_3 z)$, and $\partial m \in C(\Omega, \mathbb{R}^9)$ by $\partial m = (\partial_1 m, \partial_2 m, \partial_3 m)$.

All terms in $\mathcal{A}(z)$ can be estimated by using one of the functions G_i. For instance,

- terms like $\mu'(\theta)\partial_i\theta\partial_i u$ can be estimated by using G_1 with $\phi_1(z) = \mu'(\theta)$ and $\phi_2(\partial z) = \partial_i\theta\partial_i u$;
- terms like $\mu(\theta)\partial_{ii} u$ can be estimated by using G_2 with $\phi_1(z) = \mu(\theta)$;

- the term $K(z):\nabla^2\theta = K_{ij}(z)\partial_{ij}\theta$ can be estimated by using G_2 with $\phi_1(z) = K_{ij}(z)$;
- terms like $\alpha(\theta)|m|^2|\nabla m|^2$ can be estimated by G_3 with $\phi_1(z) = \alpha(\theta)|m|^2$ and $\phi_3(\partial m) = |\nabla m|^2$;
- the scalar components of $(\alpha(\theta)I_3 - \beta(\theta)M(m))\Delta m$ can be estimated by using G_4 with $\phi_3 \equiv 1$ and ϕ_1 properly chosen;
- the term $\alpha(\theta)|\nabla m|^2\; m\cdot\Delta m$, appearing in the θ-equation, can be estimated by using G_4 with $\phi_3(\partial m) = |\nabla m|^2$ and ϕ_1 properly chosen;
- lastly, the term $\alpha(\theta)|\Delta m|^2$ can be estimated by using G_5 with $\phi_1(z) = \alpha(\theta)$.

Lemma B.2. *Let the functions G_i, $1\le i\le 5$ be given by (B.3). Then,*

$$G_1, G_2, G_5 \in C^1(\mathbb{E}_{1,\mu}(J_T), \mathbb{E}^0_{0,\mu}(J_T)), \quad G_3, G_4 \in C^1(\mathbb{E}_{1,\mu}(J_T), \mathbb{E}^1_{0,\mu}(J_T)),$$

where $\mathbb{E}^k_{0,\mu}(J_T) = L_{p,\mu}(J_T; W^k_p(\Omega))$, $k = 0, 1$. Furthermore, given $T_0, R_0 > 0$, then for any $T \in (0, T_0]$, $R \in (0, R_0]$ and any $z_1 = (u_1, F_1, \theta_1, m_1) \in \mathbb{E}_{1,\mu}(J_T)$ and $z_2 = (u_2, F_2, \theta_2, m_2) \in {}_0\mathbb{E}_{1,\mu}(J_T)$ satisfying

$$\|z_1\|_{\mathbb{B}_\mu(J_T)},\; \|z_1\|_{\mathbb{E}_{1,\mu}(J_T)},\; \|z_2\|_{\mathbb{E}_{1,\mu}(J_T)} \le R,$$

the following estimate holds

$$\begin{aligned}
&\|G_i(z_1+z_2) - G_i(z_1) - G_i'(z_1)z_2\|_{\mathbb{E}^0_{0,\mu}(J_T)} \le \Phi(\|z_2\|_{\mathbb{E}_{1,\mu}(J_T)})\|z_2\|_{\mathbb{E}_{1,\mu}(J_T)}, \\
&\qquad i = 1, 2, 5, \\
&\|G_i(z_1+z_2) - G_i(z_1) - G_i'(z_1)z_2\|_{\mathbb{E}^1_{0,\mu}(J_T)} \le \Phi(\|z_2\|_{\mathbb{E}_{1,\mu}(J_T)})\|z_2\|_{\mathbb{E}_{1,\mu}(J_T)}, \\
&\qquad i = 3, 4,
\end{aligned} \tag{B.4}$$

where G_i' is the Frechét derivative of G_i.

Proof. The continuous differentiability of G_i follows by direct computations. We will only establish the estimates in (B.4). Easy computations lead to

$$\begin{aligned}
&G_1(z_1+z_2) - G_1(z_1) - G_1'(z_1)z_2 \\
&\quad = \big(\phi_1(z_1+z_2) - \phi_1(z_1) - \phi_1'(z_1)z_2\big)\,\phi_2(\partial z_1) \\
&\qquad + \phi_1(z_1+z_2)\,\big(\phi_2(\partial z_1 + \partial z_2) - \phi_2(\partial z_1) - \phi_2'(\partial z_1)\partial z_2\big) \\
&\qquad + (\phi_1(z_1+z_2) - \phi_1(z_1))\,\phi_2'(\partial z_1)\partial z_2.
\end{aligned}$$

Then the mean value theorem, Lemma 3.1(a), (3.3) implies

$$\begin{aligned}
&\big\|\big(\phi_1(z_1+z_2) - \phi_1(z_1) - \phi_1'(z_1)z_2\big)\,\phi_2(\partial z_1)\big\|_{\mathbb{E}^0_{0,\mu}(J_T)} \\
&\quad \le \|\phi_2(\partial z_1)\|_\infty\, \|z_2\|_\infty \int_0^1 \big\|\phi_1'(z_1+\sigma z_2) - \phi_1'(z_1)\big\|_{\mathbb{E}^0_{0,\mu}(J_T)}\, d\sigma
\end{aligned}$$

$$
\begin{aligned}
&\leq \|\phi_2(\partial z_1)\|_\infty \ \|z_2\|_\infty \int_{[0,1]\times[0,1]} \left\|\phi_1''(z_1+\tau\sigma z_2)\right\|_\infty \ \|z_2\|_{\mathbb{E}^0_{0,\mu}(J_T)} \, d\sigma \, d\tau \\
&\leq \Phi(\|z_2\|_{\mathbb{E}_{1,\mu}(J_T)})\|z_2\|_{\mathbb{E}_{1,\mu}(J_T)}.
\end{aligned}
$$

In the above, $\phi'(z)$ denotes the Frechét derivative of the Nemyskii operator induced by ϕ and we have used the fact that $\phi'(z) = \sum_{j=1}^{16} \partial_j \phi(z) \otimes e_j$, where $\partial_j \phi$ is the partial derivative of ϕ. We will take advantage of this observation in the sequel. Note that the function Φ above is uniform with respect to $T \in (0, T_0]$ in view of Lemma 3.1(a). Estimating in the same way, we have

$$
\begin{aligned}
&\|\phi_1(z_1+z_2)\left(\phi_2(\partial z_1+\partial z_2)-\phi_2(\partial z_1)-\phi_2'(\partial z_1)\partial z_2\right)\|_{\mathbb{E}^0_{0,\mu}(J_T)} \\
&\quad \leq \Phi(\|z_2\|_{\mathbb{E}_{1,\mu}(J_T)})\|z_2\|_{\mathbb{E}_{1,\mu}(J_T)}.
\end{aligned}
$$

The remaining terms can be estimated again by using the mean value theorem as follows:

$$
\begin{aligned}
&\| \left(\phi_1(z_1+z_2)-\phi_1(z_1)\right)\phi_2'(\partial z_1)\partial z_2\|_{\mathbb{E}^0_{0,\mu}(J_T)} \\
&\quad \leq \|\phi_2'(\partial z_1)\|_\infty \ \|\partial z_2\|_\infty \int_0^1 \left(\|\phi_1'(z_1+\sigma z_2)\|_\infty \ \|z_2\|_{\mathbb{E}^0_{0,\mu}(J_T)}\right) d\sigma \\
&\quad \leq \Phi(\|z_2\|_{\mathbb{E}_{1,\mu}(J_T)})\|z_2\|_{\mathbb{E}_{1,\mu}(J_T)}.
\end{aligned}
$$

The estimate for G_3 and G_5 can be obtained in the same manner in view of the additional regularity of m.

The estimate for G_2 will be slightly different in the sense that we need to evaluate $\partial_{ij} z_k$, $k = 1, 2$, by using the $\mathbb{E}^0_{0,\mu}(J_T)$-norm. First, notice that

$$
\begin{aligned}
&G_2(z_1+z_2)-G_2(z_1)-G_2'(z_1)z_2, \\
&\quad = \left(\phi_1(z_1+z_2)-\phi_1(z_1)-\phi_1'(z_1)z_2\right)\partial_{ij}z_1 + \left(\phi_1(z_1+z_2)-\phi_1(z_1)\right)\partial_{ij}z_2.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\| \left(\phi_1(z_1+z_2)-\phi_1(z_1)-\phi_1'(z_1)z_2\right)\partial_{ij}z_1\|_{\mathbb{E}^0_{0,\mu}(J_T)} \\
&\quad \leq \|\partial_{ij}z_1\|_{\mathbb{E}^0_{0,\mu}(J_T)} \ \|z_2\|_\infty \int_0^1 \left\|\phi_1'(z_1+\sigma z_2)-\phi_1'(z_1)\right\|_\infty d\sigma \\
&\quad \leq \Phi(\|z_2\|_{\mathbb{E}_{1,\mu}(J_T)})\|z_2\|_{\mathbb{E}_{1,\mu}(J_T)}.
\end{aligned}
$$

Similarly,

$$
\| \left(\phi_1(z_1+z_2)-\phi_1(z_1)\right)\partial_{ij}z_2\|_{\mathbb{E}^0_{0,\mu}(J_T)} \leq \Phi(\|z_2\|_{\mathbb{E}_{1,\mu}(J_T)})\|z_2\|_{\mathbb{E}_{1,\mu}(J_T)}.
$$

The estimate for G_4 can be derived in a similar way by utilizing the additional regularity of m and the facts that

$$
\|G_6(z_1+z_2)-G_6(z_1)-G_6'(z_1)z_2\|_{C([0,T];C^1(\overline{\Omega}))} \leq \Phi(\|z_2\|_{C([0,T];C^1(\overline{\Omega}))})
$$

$$\|z_2\|_{C([0,T];C^1(\overline{\Omega}))},$$
$$\|G_6(z_1+z_2)-G_6(z_1)\|_{C([0,T];C^1(\overline{\Omega}))}\le M\|z_2\|_{C([0,T];C^1(\overline{\Omega}))},$$

where $G_6(z)=\phi_1(z)\phi_3(\partial m)$. □

We are now ready to establish the differentiability of $(\mathsf{A},\mathsf{B},\mathsf{F})$ as operators defined on $\mathbb{E}_{1,\mu}(J_T)$.

Proposition B.3. *Assume* (2.1) *and* (3.2). *Then,*

$$\begin{aligned}
&\mathcal{A}\in C^1(\mathbb{E}_{1,\mu}(J_T),\mathbb{E}_{0,\mu}(J_T)), && \mathcal{A}'(z_*)z=\mathsf{A}(z_*)z+[\mathsf{A}'(z_*)z]z_*,\\
&\mathsf{F}\in C^1(\mathbb{E}_{1,\mu}(J_T),\mathbb{E}_{0,\mu}(J_T))), && \\
&\mathcal{B}\in C^1(\mathbb{E}_{1,\mu}(J_T),\mathbb{F}_{\mu}(J_T)), && \mathcal{B}'(z_*)z=\mathsf{B}(z_*)z+[\mathsf{B}'(z_*)z]z_*,
\end{aligned}$$

for $z_*, z\in\mathbb{E}_{1,\mu}(J_T)$*, where the mappings* $(\mathcal{A},\mathcal{B})$ *were introduced in* (4.12). *Moreover, given* $T_0, R_0>0$*, then for any* $T\in(0,T_0]$*,* $R\in(0,R_0]$ *and any* $z_*\in\mathbb{E}_{1,\mu}(J_T)$*,* $z\in{}_0\mathbb{E}_{1,\mu}(J_T)$ *satisfying*

$$\|\mathrm{tr}_{\partial\Omega}z_*\|_{\mathbb{F}_\mu(J_T)},\ \|\mathrm{tr}_{\partial\Omega}\nabla z_*\|_{\mathbb{F}_\mu(J_T)},\ \|z_*\|_{\mathbb{B}_\mu(J_T)},\ \|z_*\|_{\mathbb{E}_{1,\mu}(J_T)},\ \|z\|_{\mathbb{E}_{1,\mu}(J_T)}\le R,$$

the following estimates hold:

$$\begin{aligned}
\|\mathcal{A}(z_*+z)-\mathcal{A}(z_*)-\mathcal{A}'(z_*)z\|_{\mathbb{E}_{0,\mu}(J_T)}&\le\Phi(\|z\|_{\mathbb{E}_{1,\mu}(J_T)})\|z\|_{\mathbb{E}_{1,\mu}(J_T)},\\
\|\mathsf{F}(z_*+z)-\mathsf{F}(z_*)-\mathsf{F}'(z_*)z\|_{\mathbb{E}_{0,\mu}(J_T)}&\le\Phi(\|z\|_{\mathbb{E}_{1,\mu}(J_T)})\|z\|_{\mathbb{E}_{1,\mu}(J_T)},\\
\|\mathcal{B}(z_*+z)-\mathcal{B}(z_*)-\mathcal{B}'(z_*)z\|_{\mathbb{F}_{\mu}(J_T)}&\le\Phi(\|z\|_{\mathbb{E}_{1,\mu}(J_T)})\|z\|_{\mathbb{E}_{1,\mu}(J_T)}.
\end{aligned}\tag{B.5}$$

If, in addition, $\overline{z}\in\mathbb{E}_{1,\mu}(J_T)$ *with* $z_*(0)=\overline{z}(0)$ *satisfies*

$$\|\mathrm{tr}_{\partial\Omega}\overline{z}\|_{\mathbb{F}_\mu(J_T)},\|\mathrm{tr}_{\partial\Omega}\nabla\overline{z}\|_{\mathbb{F}_\mu(J_T)},\|\overline{z}\|_{\mathbb{E}_{1,\mu}(J_T)},\|\overline{z}\|_{\mathbb{B}_\mu(J_T)}\le R,$$

then the following estimates hold

$$\begin{aligned}
\|\mathcal{A}'(z_*)z-\mathcal{A}'(\overline{z})z\|_{\mathbb{E}_{0,\mu}(J_T)}&\le\Phi(\|z_*-\overline{z}\|_{\mathbb{E}_{1,\mu}(J_T)})\|z\|_{\mathbb{E}_{1,\mu}(J_T)},\\
\|\mathsf{F}'(z_*)z-\mathsf{F}'(\overline{z})z\|_{\mathbb{E}_{0,\mu}(J_T)}&\le\Phi(\|z_*-\overline{z}\|_{\mathbb{E}_{1,\mu}(J_T)})\|z\|_{\mathbb{E}_{1,\mu}(J_T)},\\
\|\mathcal{B}'(z_*)z-\mathcal{B}'(\overline{z})z\|_{\mathbb{F}_{\mu}(J_T)}&\le\Phi(\|z_*-\overline{z}\|_{\mathbb{E}_{1,\mu}(J_T)})\|z\|_{\mathbb{E}_{1,\mu}(J_T)}.
\end{aligned}\tag{B.6}$$

Proof. The continuous differentiability of $\mathcal{A}$ and F and the first two estimates in (B.5) are immediate consequences of Lemma B.2. The continuous differentiability of $\mathcal{B}$ is a direct consequence of Lemma B.1 and the fact that $\mathbb{F}_\mu(J_T)$ is a Banach algebra. To establish the last estimate in (B.5), we set $z=(z_j)_{j=1}^{16}=(u,F,\theta,m)$ and $z_*=(u_*,F_*,\theta_*,m_*)$. Then, we can apply (B.1), (B.2), Lemma A.5(i) and (ii), Proposition A.4, and [28, Theorem 4.5] to obtain (where we suppress $\mathrm{tr}_{\partial\Omega}$ in the following computations)

$$
\begin{aligned}
&\|\mathcal{B}(z_*+z)-\mathcal{B}(z_*)-\mathcal{B}'(z_*)z\|_{\mathbb{F}_\mu(J_T)}\\
&\quad\le \left\|\left[\left(\int_0^1 \left(K'(z_*+\sigma z)-K'(z_*)\right)\, d\sigma\right) z\right]\nabla\theta_*\right\|_{\mathbb{F}_\mu(J_T)}\\
&\qquad+\left\|\left[\left(\int_0^1 \left(K'(z_*+\sigma z)\right)\, d\sigma\right) z\right]\nabla\theta\right\|_{\mathbb{F}_\mu(J_T)}\\
&\quad\le C\left\|\int_0^1 \left(K'(z_*+\sigma z)-K'(z_*)\right)\, d\sigma\right\|_{\mathbb{F}_\mu(J_T)} \|\nabla\theta_*\|_{\mathbb{F}_\mu(J_T)}\ \|z\|_{\mathbb{F}_{1,\mu}(J_T)}\\
&\qquad+C\left\|\int_0^1 K'(z_*+\sigma z)\, d\sigma\right\|_{\mathbb{F}_\mu(J_T)} \|\nabla\theta\|_{\mathbb{F}_\mu(J_T)}\ \|z\|_{\mathbb{F}_{1,\mu}(J_T)}\\
&\quad\le \Phi(\|z\|_{\mathbb{E}_{1,\mu}(J_T)})\|z\|_{\mathbb{E}_{1,\mu}(J_T)}.
\end{aligned}
$$

This establishes the last estimate in (B.5).

Concerning the estimates in (B.6), we will only establish the last one. The remaining two follow from a similar argument.

$$
\begin{aligned}
\|\mathcal{B}'(z_*)z-\mathcal{B}'(\overline{z})z\|_{\mathbb{F}_\mu(J_T)} &= \|\mathsf{B}(z_*)z-\mathsf{B}(\overline{z})z-[\mathsf{B}'(z_*)z]z_*+[\mathsf{B}'(\overline{z})z]\overline{z}\|_{\mathbb{F}_\mu(J_T)}\\
&\le \|\mathsf{B}(z_*)z-\mathsf{B}(\overline{z})z\|_{\mathbb{F}_\mu(J_T)}+\|[\mathsf{B}'(z_*)z]z_*-[\mathsf{B}'(\overline{z})z]\overline{z}\|_{\mathbb{F}_\mu(J_T)}.
\end{aligned}
$$

Let $\overline{z}=(\overline{u},\overline{F},\overline{\theta},\overline{m})$. Then, the first term on the right-hand side can be estimated by using (B.1), (B.2), Lemma A.5(i) and (ii), Proposition A.4, and [28, Theorems 4.2 and 4.5] as follows:

$$
\begin{aligned}
\|\mathsf{B}(z_*)z-\mathsf{B}(\overline{z})z\|_{\mathbb{F}_\mu(J_T)} &\le C\|K(z_*)-K(\overline{z})\|_{\mathbb{F}_\mu(J_T)}\|\nabla\theta\|_{\mathbb{F}_\mu(J_T)}\\
&\le C\int_0^1 \|K'(\sigma z_*+(1-\sigma)\overline{z})\|_{\mathbb{F}_\mu(J_T)}\, d\sigma\,\|z_*\\
&\quad-\overline{z}\|_{\mathbb{F}_{1,\mu}(J_T)}\|z\|_{\mathbb{E}_{1,\mu}(J_T)}\\
&\le \Phi(\|z_*-\overline{z}\|_{\mathbb{E}_{1,\mu}(J_T)})\|z\|_{\mathbb{E}_{1,\mu}(J_T)}.
\end{aligned}
$$

The estimate of the second term can be obtained analogously:

$$
\begin{aligned}
&\|[\mathsf{B}'(z_*)z]z_*-[\mathsf{B}'(\overline{z})z]\overline{z}\|_{\mathbb{F}_\mu(J_T)}\\
&\quad\le \|[K'(z_*)z]\nabla(\theta_*-\overline{\theta})\|_{\mathbb{F}_\mu(J_T)}+\|[(K'(z_*)-K'(\overline{z}))z]\nabla\overline{\theta}\|_{\mathbb{F}_\mu(J_T)}\\
&\quad\le C\left(\|K'(z_*)z\|_{\mathbb{F}_\mu(J_T)}\|\nabla(\theta_*-\overline{\theta})\|_{\mathbb{F}_\mu(J_T)}+\sum_{j=1}^{16}\|\left(\partial_j K(z_*)-\partial_j K(\overline{z})\right)\|_{\mathbb{F}_\mu(J_T)}\|z_j\nabla\overline{\theta}\|_{\mathbb{F}_\mu(J_T)}\right)\\
&\quad\le C\left(\|K'(z_*)\|_{\mathbb{F}_\mu(J_T)}+\Phi(R)\|\nabla\overline{\theta}\|_{\mathbb{F}_\mu(J_T)}\right)\|z\|_{\mathbb{F}_{1,\mu}(J_T)}\|z_*-\overline{z}\|_{\mathbb{E}_{1,\mu}(J_T)}\\
&\quad\le \Phi(\|z_*-\overline{z}\|_{\mathbb{E}_{1,\mu}(J_T)})\|z\|_{\mathbb{E}_{1,\mu}(J_T)}.
\end{aligned}
$$

□

To study the continuous dependence of solutions to (3.7) on the initial data, see Theorem 5.1(b), we need the following result.

Lemma B.3. *Let $\mathcal{B}$ be as in (4.12). Then, we have*

(a) $\mathcal{B} \in C^1(X_{\gamma,\mu}, Y_{\gamma,\mu})$ *and* $\mathcal{B}'(z_0)z = \mathsf{B}(z_0)z + [\mathsf{B}'(z_0)z]z_0, \quad z_0, z \in X_{\gamma,\mu}$.

(b) *For each* $z_0 \in X_{\gamma,\mu}$, $\mathcal{B}'(z_0) \in \mathcal{L}(X_{\gamma,\mu}, Y_{\gamma,\mu})$ *has a bounded right inverse* $\mathcal{R}(z_0) \in \mathcal{L}(Y_{\gamma,\mu}, X_{\gamma,\mu})$.

Proof. (a) By Lemma 3.1, the trace operator $\gamma_0 \in \mathcal{L}(\mathbb{E}_{1,\mu}(J_T), X_{\gamma,\mu})$ has a right inverse $\gamma_0^c \in \mathcal{L}(X_{\gamma,\mu}, \mathbb{E}_{1,\mu}(J_T))$. It is then easy to see that $\mathcal{B}(z) = \widetilde{\gamma}_0\mathcal{B}(\gamma_0^c(z))$, where $\widetilde{\gamma}_0$ denotes the initial time trace operator for functions defined on $\mathbb{F}_\mu(J_T)$. The assertions then follow from Proposition B.3.

(b) The existence of $\mathcal{R}(z_0)$ is proved in [26, Proposition 2.5.1]. □

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