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Well-posedness for magnetoviscoelastic fluids in 3D☆

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ABSTRACT

We show that the system of equations describing a magnetoviscoelastic fluid in three dimensions can be cast as a quasilinear parabolic system. Using the theory of maximal L_p-regularity, we establish existence and uniqueness of local strong solutions and we show that each solution is smooth (in fact analytic) in space and time. Moreover, we give a complete characterization of the set of equilibria and show that solutions that start out close to a constant equilibrium exist globally and converge to a (possibly different) constant equilibrium. Finally, we show that every solution that is eventually bounded in the topology of the state space exists globally and converges to the set of equilibria.

1. Introduction

We will study the following system of equations that models the evolution of a magnetoviscoelastic fluid

$$\begin{cases} \partial_t u + u\cdot\nabla u - \mu_s \Delta u + \nabla\pi = -\nabla\cdot(\nabla m \odot \nabla m) + \nabla\cdot(FF^{\mathsf{T}}) & \text{in } \mathbb{R}_+\times\Omega, \\ \nabla\cdot u = 0 & \text{in } \mathbb{R}_+\times\Omega, \\ u = 0 & \text{on } \mathbb{R}_+\times\partial\Omega, \\ \partial_t F + u\cdot\nabla F - (\nabla u)^{\mathsf{T}} F = \kappa\Delta F & \text{in } \mathbb{R}_+\times\Omega, \\ F = 0 & \text{on } \mathbb{R}_+\times\partial\Omega, \\ \partial_t m + u\cdot\nabla m = -\alpha m\times(m\times\Delta m) - \beta m\times\Delta m & \text{in } \mathbb{R}_+\times\Omega, \\ \partial_\nu m = 0 & \text{on } \mathbb{R}_+\times\partial\Omega, \\ |m| = 1 & \text{in } \mathbb{R}_+\times\Omega, \\ (u(0),F(0),m(0)) = (u_0,F_0,m_0) & \text{in } \Omega. \end{cases} \tag{1.1}$$

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Here, $\Omega \subset \mathbb{R}^3$ is a bounded connected C^3-domain with outward unit normal field ν. The unknowns $(u, F, m) : \mathbb{R}_+ \times \Omega \to \mathbb{R}^3 \times \mathbb{M}^3 \times \mathbb{R}^3$ denote the fluid velocity, the deformation tensor field and the magnetization field, respectively, while π is the pressure. Moreover, $\mathbb{M}^3$ stands for the set of all (3×3)-real matrices. The parameters $\alpha, \beta > 0$ are the so-called Gilbert damping and the exchange constant, while μ_s and κ are the dynamic viscosity and dissipative coefficient, respectively.
The notation $\nabla m \odot \nabla m$ means $\nabla m (\nabla m)^{\mathsf{T}}$. Hence, $\nabla m \odot \nabla m$ is a symmetric tensor with coefficients $[\nabla m \odot \nabla m]_{ij} = \partial_i m \cdot \partial_j m$.

(1.1) is a coupled system of equations containing

- the incompressible Navier–Stokes equations for the velocity field u and including in addition magnetic and elastic terms in the stress tensor,
- a transport-stretch-dissipative system for the deformation tensor F,
- a convected Landau–Lifshitz–Gilbert system for the magnetization field m.

As a multi-physical hydrodynamics model, (1.1) enjoys the following energy dissipation property:

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\left(|u|^2 + |F|^2 + |\nabla m|^2\right)\, dx = -\int_\Omega \left(\mu_s|\nabla u|^2 + \kappa|\nabla F|^2 + \alpha|\,\Delta m + |\nabla m|^2 m\,|^2\right)\, dx,$$

see Proposition 4.1.

The system (1.1) was first introduced in [1,2]. This model can describe the motion of fluids with micromagnetic and elastic particles such as ferrofluids [3,4] and magnetorheological fluids [5]. Existence of weak solutions in 2D was established in [1] under a smallness condition on the initial data by using a Galerkin approximation. In [6], the authors extended these results under more general assumptions on the elastic energy density. Moreover, they proved local-in-time existence of strong solutions and they established a weak-strong uniqueness property. Recently, the authors in [7] obtained global weak solutions to (1.1) with partial regularity in a 2D periodic domain by a careful blow-up analysis near singularities.

The main difficulty in constructing global weak solutions to (1.1) is caused by the lack of sufficient integrability in the a priori energy estimates for the stress tensor term $\nabla m \odot \nabla m$.

In 3D, the authors in [2,8,9] consider the simplified system

$$\begin{cases} \partial_t u + u\cdot\nabla u - \mu_s \Delta u + \nabla\pi = -\nabla\cdot(\nabla m \odot \nabla m) + \nabla\cdot(FF^{\mathsf{T}}), \\ \nabla\cdot u = 0, \\ \partial_t F + u\cdot\nabla F - (\nabla u)^{\mathsf{T}} F = \kappa\Delta F, \\ \partial_t m + u\cdot\nabla m = \Delta m - \dfrac{1}{\varepsilon^2}(|m|^2-1)m, \end{cases} \tag{1.2}$$

where the constraint $|m| \equiv 1$ is replaced by the Ginzburg–Landau penalization term $\frac{1}{\varepsilon^2}(|m|^2-1)^2$.

In [2], the author adapted the approach in [10] to show the existence of weak solutions to (1.2) with the combination of a Galerkin approximation scheme and a fixed point argument. The weak-strong uniqueness of solutions to (1.2) was established in [9] under the Prodi–Serrin condition. In case the initial values have higher regularity, the authors in [8] obtained the well-posedness of strong solutions to (1.2) via a priori estimates that are uniform in the approximate solutions.

We would like to point out that the regularization term $\kappa\Delta F$ in (1.1) and (1.2) with $0 < \kappa \ll 1$ plays an important role in the mathematical analysis. If $\kappa = 0$, the evolution of the deformation tensor field becomes hyperbolic, and in this case, even in 2D, the existence of weak solutions to incompressible viscoelastic fluids ($m = 0$) with large initial data remains an open problem. Local well-posedness of strong solutions to (1.2) without the regularization term was established in [11] in a periodic domain in two or three dimensions.

From the viewpoint of modeling, $F = 0$ represents a liquid phase that contains no elastic solid particles. We refer the reader to [12] for more details.

To the best of our knowledge, there are so far no existence results for system (1.1) in 3D. In our approach, we consider (1.1) as a quasilinear system and prove that the system is parabolic. We can then apply the theory of maximal L_p- regularity to establish short time existence and uniqueness of strong solutions, see Theorem 2.5. In Sections 3 and 4, we show that the set of equilibria of (1.1) is given by

$$\mathcal{E} = \{(0, 0_3, m_*, \pi_*)\},$$

where $m_* \in H^2_q(\Omega, \mathbb{R}^3)$ solves the nonlinear constrained elliptic problem

$$\begin{cases} \Delta m_* + |\nabla m_*|^2 m_* = 0 & \text{in} \quad \Omega, \\ |m_*| \equiv 1 & \text{in} \quad \Omega, \\ \partial_\nu m_* = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.3}$$

and $\pi_* = -\frac{1}{2}|\nabla m_*|^2 + C$ for some constant C.

In particular, we have that

$$\mathcal{E}_c := \{(0, 0_3, m_*, \pi_*) \in \mathbb{R}^3 \times \mathbb{M}^3 \times \mathbb{S}^2 \times \mathbb{R}\} \subset \mathcal{E}.$$

We call $\mathcal{E}_c$ the set of constant equilibria. We can prove that all constant equilibria are normally stable, and that each solution that starts out close to a constant equilibrium exists globally and converges to a (possibly different) constant equilibrium. Moreover, we show that any solution that is bounded in an appropriate topology exists globally and converges to the set $\mathcal{E}$ of equilibria.

In case we choose $(u_0, F_0) = (0, 0)$, system (1.1) reduces to the well-known Landau–Lifshitz–Gilbert equation

$$\begin{cases} \partial_t m = -\alpha m \times (m \times \Delta m) - \beta m \times \Delta m & \text{in} \quad \mathbb{R}_+ \times \Omega, \\ \partial_\nu m = 0 & \text{on} \quad \mathbb{R}_+ \times \partial\Omega, \\ |m| = 1 & \text{in} \quad \mathbb{R}_+ \times \Omega, \\ m(0) = m_0 & \text{in} \quad \Omega. \end{cases} \tag{1.4}$$

In this case, we obtain the energy dissipation relation

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|\nabla m|^2\, dx = -\int_\Omega \alpha |\, \Delta m + |\nabla m|^2 m\,|^2\, dx.$$

By the same arguments as in Section 4, we can conclude that the set of equilibria of (1.4) is given by the solutions of (1.3). Hence, all the results established for system (1.1) remain true for the Landau–Lifshitz–Gilbert equation. A similar result was obtained in [13] in case $\Omega = \mathbb{R}^n$ with $n \geq 3$.

Finally, we mention that all of our results remain valid in 2D, that is, in case $\Omega \subset \mathbb{R}^2$ and $(u, F, m) : \Omega \to \mathbb{R}^2 \times \mathbb{M}^2 \times \mathbb{R}^3$.

Notation: For the readers' convenience, we list here some notation and conventions used throughout the manuscript.

In the following, all vectors $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ are viewed as column vectors. For two vectors $a, b \in \mathbb{R}^n$, the Euclidean inner product is denoted by $a \cdot b$. Given two matrices $A, B \in \mathbb{M}^n$, the Frobenius matrix inner product $A : B$ is given by

$$A : B = \mathrm{Tr}(AB^\mathsf{T}),$$

where $^\mathsf{T}$ is the transpose. Suppose Ω is an open subset of $\mathbb{R}^n$. If $u \in C^1(\Omega; \mathbb{R}^n)$, we set $\nabla u(x) = e_j \otimes \partial_j u(x)$ for $x \in \Omega$. Hence, for $u = (u_1, \ldots, u_n) \in C^1(\Omega; \mathbb{R}^n)$, we have

$$[\nabla u(x)]_{ij} = \partial_i u_j(x), \quad 1 \leq i, j \leq n, \quad x \in \Omega. \tag{1.5}$$

We note that $[\nabla u(x)]^\mathsf{T}$ corresponds to the Fréchet derivative of u at $x \in \Omega$.

If $A \in C^1(\Omega; \mathbb{M}^n)$, its divergence $\nabla \cdot A$ is the vector function defined by

$$(\nabla \cdot A)(x) = (\partial_j A(x))^{\mathsf{T}} e_j, \quad x \in \Omega. \tag{1.6}$$

Hence, if $A = [a_{ij}] \in C^1(\Omega; \mathbb{M}^n)$, its divergence is given by

$$[(\nabla \cdot A)(x)]_i = \partial_j a_{ji}(x), \quad i = 1, \ldots, n, \quad x \in \Omega.$$

Here and in the sequel, we use the summation convention, indicating that terms with repeated indices are added. We note that (1.5) and (1.6) imply

$$\nabla \cdot (\nabla u) = \Delta u, \quad u \in C^2(\Omega; \mathbb{R}^n),$$

and

$$(\nabla \cdot A) \cdot u = \nabla \cdot (Au) - A : \nabla u, \quad A \in C^1(\Omega; \mathbb{M}^n),\ u \in C^1(\Omega; \mathbb{R}^n). \tag{1.7}$$

For a matrix $A \in C^1(\Omega; \mathbb{M}^n)$, we set $|\nabla A|^2 = \partial_j A : \partial_j A$.

For functions $f, g \in L_2(\Omega; \mathbb{R}^m)$,

$$(f|g)_\Omega = \int_\Omega f \cdot g \, dx$$

denotes the L_2-inner product. For any Banach space X, $s \geq 0$, $p, q \in (1, \infty)$, $B^s_{pq}(\Omega; X)$ denote the $X-$valued Besov spaces, whereas $H^s_q(\Omega; X)$ are the Bessel-potential spaces. When the choice of X is clear from the context, we will just write $B^s_{pq}(\Omega)$ or $H^s_q(\Omega)$.

For $p \in (1, \infty)$ and $\mu \in (0, 1]$, the X-valued L_p-spaces with temporal weight are defined by

$$L_{p,\mu}((0,T); X) := \left\{ f : (0,T) \to X : \, t^{1-\mu} f(t) \in L_p((0,T); X) \right\}.$$

Similarly,

$$H^1_{p,\mu}((0,T); X) := \left\{ f \in L_{p,\mu}((0,T); X) \cap H^1_1((0,T); X) : \, f'(t) \in L_{p,\mu}((0,T); X) \right\}.$$

For any two Banach spaces X and Y, the notation $\mathcal{L}(X, Y)$ stands for the set of all bounded linear maps from X to Y and $\mathcal{L}(X) := \mathcal{L}(X, X)$.

2. Existence and uniqueness of solutions

In this section, we show how to formulate system (1.1) as a quasilinear equation. Using the theory of maximal L_p-regularity, we establish existence and uniqueness of local in time solutions, and we show that solutions have additional time regularity. We start by expressing the term

$$\alpha m \times (m \times \Delta m) + \beta m \times \Delta m$$

in a form that is more convenient for our analysis. By the well-known identity $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$, we have

$$m \times (m \times \Delta m) = (m \cdot \Delta m)m - |m|^2 \Delta m.$$

By using the facts that $|m| = 1$ and $0 = \Delta |m|^2 = 2|\nabla m|^2 + 2\, m \cdot \Delta m$, we obtain

$$m \times (m \times \Delta m) = -(\Delta m + |\nabla m|^2 \, m), \tag{2.1}$$

provided m is sufficiently smooth. Setting

$$\mathsf{M}(m) = \begin{bmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{bmatrix}, \quad m = (m_1, m_2, m_3),$$

we can write $m \times \Delta m = \mathsf{M}(m)\Delta m$. Hence, under the constraint $|m| \equiv 1$, (1.1) is equivalent to the following system

$$\begin{cases} \partial_t u + u\cdot\nabla u - \mu_s \Delta u + \nabla\pi = -\nabla\cdot(\nabla m \odot \nabla m) + \nabla\cdot(FF^{\mathsf{T}}) & \text{in} \quad \mathbb{R}_+ \times \Omega, \\ \nabla\cdot u = 0 & \text{in} \quad \mathbb{R}_+ \times \Omega, \\ u = 0 & \text{on} \quad \mathbb{R}_+ \times \partial\Omega, \\ \partial_t F - \kappa\Delta F = (\nabla u)^{\mathsf{T}} F - u\cdot\nabla F & \text{in} \quad \mathbb{R}_+ \times \Omega, \\ F = 0 & \text{on} \quad \mathbb{R}_+ \times \partial\Omega, \\ \partial_t m + u\cdot\nabla m = (\alpha I_3 - \beta\mathsf{M}(m))\Delta m + \alpha|\nabla m|^2\, m & \text{in} \quad \mathbb{R}_+ \times \Omega, \\ \partial_\nu m = 0 & \text{on} \quad \mathbb{R}_+ \times \partial\Omega, \\ |m| = 1 & \text{in} \quad \mathbb{R}_+ \times \Omega, \\ (u(0), F(0), m(0)) = (u_0, F_0, m_0) & \text{in} \quad \Omega, \end{cases} \tag{2.2}$$

where I_3 is the 3×3 identity matrix. Neglecting the constraint $|m| \equiv 1$, we have

$$\begin{cases} \partial_t u + u\cdot\nabla u - \mu_s \Delta u + \nabla\pi = -\nabla\cdot(\nabla m \odot \nabla m) + \nabla\cdot(FF^{\mathsf{T}}) & \text{in} \quad \mathbb{R}_+ \times \Omega, \\ \nabla\cdot u = 0 & \text{in} \quad \mathbb{R}_+ \times \Omega, \\ u = 0 & \text{on} \quad \mathbb{R}_+ \times \partial\Omega, \\ \partial_t F - \kappa\Delta F = (\nabla u)^{\mathsf{T}} F - u\cdot\nabla F & \text{in} \quad \mathbb{R}_+ \times \Omega, \\ F = 0 & \text{on} \quad \mathbb{R}_+ \times \partial\Omega, \\ \partial_t m + u\cdot\nabla m = (\alpha I_3 - \beta\mathsf{M}(m))\Delta m + \alpha|\nabla m|^2\, m & \text{in} \quad \mathbb{R}_+ \times \Omega, \\ \partial_\nu m = 0 & \text{on} \quad \mathbb{R}_+ \times \partial\Omega, \\ (u(0), F(0), m(0)) = (u_0, F_0, m_0) & \text{in} \quad \Omega. \end{cases} \tag{2.3}$$

We will first study the unconstrained system (2.3), and then show in a second step that the constraint $|m| \equiv 1$ is preserved in case $|m_0| \equiv 1$.

The main tool to study (2.3) is the theory of maximal L_p-regularity. For $\theta \in (0, \pi]$, the open sector with angle 2θ is denoted by

$$\Sigma_\theta := \{\omega \in \mathbb{C} \setminus \{0\} : |\arg\omega| < \theta\}.$$

Definition 2.1. Let X be a complex Banach space, and $\mathcal{A}$ be a densely defined closed linear operator in X with dense range. $\mathcal{A}$ is called sectorial if $\Sigma_\theta \subset \rho(-\mathcal{A})$ for some $\theta > 0$ and

$$\sup\{\|\lambda(\lambda + \mathcal{A})^{-1}\|_{\mathcal{L}(X)} : \lambda \in \Sigma_\theta\} < \infty.$$

The class of sectorial operators in X is denoted by $\mathcal{S}(X)$.

To introduce the notion of maximal L_p-regularity, let us consider the following abstract Cauchy problem on $[0, T]$

$$\begin{cases} \partial_t u(t) + \mathcal{A}u(t) = f(t), \quad t \in (0, T), \\ u(0) = 0. \end{cases} \tag{2.4}$$

Definition 2.2. Assume that $X_1 \overset{d}{\hookrightarrow} X_0$ is some densely embedded Banach couple. Suppose that $\mathcal{A} \in \mathcal{S}(X_0)$ with $D(\mathcal{A}) = X_1$. The operator $\mathcal{A}$ is said to have the property of maximal L_p-regularity if for any fixed $T > 0$ and

$$f \in L_p((0, T); X_0),$$

(2.4) has a unique solution

$$u \in L_p((0,T);X_1) \cap H^1_p((0,T);X_0).$$

We denote the set of all operators $\mathcal{A} \in S(X)$ which enjoy the property of maximal L_p-regularity by

$$\mathcal{A} \in \mathcal{MR}_p(X_1, X_0).$$

We refer to [14] for additional background information.

Let $P_H : L_q(\Omega;\mathbb{R}^3) \to L_{q,\sigma}(\Omega;\mathbb{R}^3)$ be the Helmholtz projection, where

$$L_{q,\sigma}(\Omega;\mathbb{R}^3) := P_H(L_q(\Omega;\mathbb{R}^3))$$

is the space of all solenoidal vector fields in $L_q(\Omega;\mathbb{R}^3)$. Setting

$$H^2_{q,\sigma}(\Omega;\mathbb{R}^3) := H^2_q(\Omega;\mathbb{R}^3) \cap L_{q,\sigma}(\Omega;\mathbb{R}^3),$$

we let $\mathcal{A}_q : D(\mathcal{A}_q) \to L_{q,\sigma}(\Omega;\mathbb{R}^3)$ be the Stokes operator, defined by

$$\mathcal{A}_q u := -\mu_s P_H \Delta u, \quad D(\mathcal{A}_q) := \{u \in H^2_{q,\sigma}(\Omega;\mathbb{R}^3) : u = 0 \text{ on } \partial\Omega\}.$$

Similarly, we can define $\mathcal{G}_q : D(\mathcal{G}_q) \to L_q(\Omega;\mathbb{M}^3)$ by

$$\mathcal{G}_q F := -\kappa \Delta F, \quad D(\mathcal{G}_q) := \{F \in H^2_q(\Omega;\mathbb{M}^3) : F = 0 \text{ on } \partial\Omega\}.$$

Further, given any $m \in C(\overline{\Omega};\mathbb{R}^3)$, the operator $\mathcal{D}_q(m) : D(\mathcal{D}_q(m)) \to L_q(\Omega;\mathbb{R}^3)$ is defined by

$$\begin{aligned} &\mathcal{D}_q(m)h := -(\alpha I_3 - \beta \mathsf{M}(m))\Delta h, \\ &D(\mathcal{D}_q(m)) := \{h \in H^2_q(\Omega;\mathbb{R}^3) : \partial_\nu h = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

Next, we set

$$[\mathcal{B}_q(m)h]_i = \partial_i m \cdot \Delta h + \nabla m : \partial_i \nabla h, \quad i = 1,2,3. \tag{2.5}$$

It follows that

$$\mathcal{B}_q(m)m = \nabla \cdot (\nabla m \odot \nabla m) \quad \text{for each } m \in H^2_q(\Omega;\mathbb{R}^3). \tag{2.6}$$

Note that $\mathcal{B}_q(m) \in \mathcal{L}(D(\mathcal{D}_q(m)), L_q(\Omega;\mathbb{R}^3))$ for any $m \in C^1(\overline{\Omega};\mathbb{R}^3)$. Finally, we define the spaces

$$X_0 = L_{q,\sigma}(\Omega;\mathbb{R}^3) \times L_q(\Omega;\mathbb{M}^3) \times L_q(\Omega;\mathbb{R}^3)$$

and

$$X_1 = D(\mathcal{A}_q) \times D(\mathcal{G}_q) \times D(\mathcal{D}_q(m)).$$

It is well known that $\mathcal{A}_q$ and $\mathcal{G}_q$ enjoy the property of maximal L_p-regularity, cf. [15–17] and [14, Section 6.3, Chapter 7]. To deal with $\mathcal{D}_q(m)$ for $m \in C(\overline{\Omega};\mathbb{R}^3)$, we set

$$\mathcal{D}_q(m(x),\xi) := (\alpha I_3 - \beta \mathsf{M}(m(x)))|\xi|^2, \quad x \in \Omega, \ \xi \in \mathbb{R}^3,$$

for the symbol of the differential operator $\mathcal{D}_q(m)$. An easy computation shows that

$$\sigma(\mathcal{D}_q(m(x),\xi)) = \{\alpha, \alpha \pm i\beta|m(x)|\}, \quad x \in \Omega, \ \xi \in \mathbb{S}^2,$$

where σ denotes the spectrum. Since $m \in C(\overline{\Omega};\mathbb{R}^3)$, $\mathcal{D}_q(m(x),\xi)$ is normally elliptic for every $x \in \overline{\Omega}$, see for instance [14, Definition 6.1.1]. By [14, Theorem 6.3.2], $\mathcal{D}_q(m)$ has the property of maximal L_p-regularity. Then, the operator $A_q : X_1 \to X_0$ defined by

$$A_q(m) = \begin{bmatrix} \mathcal{A}_q & 0 & P_H \mathcal{B}_q(m) \\ 0 & \mathcal{G}_q & 0 \\ 0 & 0 & \mathcal{D}_q(m) \end{bmatrix} \tag{2.7}$$

enjoys the property of maximal L_p-regularity for every $m \in C^1(\overline{\Omega};\mathbb{R}^3)$ as well, due to its upper triangular structure.

Indeed, given any $f=(f_1,f_2,f_3) \in L_p((0,T);X_0)$, we consider the system

$$\begin{cases} \partial_t z + A_q(m)z = f(t), \quad t \in (0,T), \\ z(0)=0, \end{cases} \tag{2.8}$$

where $z=(v,G,h)$. By the maximal L_p-regularity property of $\mathcal{G}_q$ and $\mathcal{D}_q(m)$, one can find for each $m \in C(\overline{\Omega};\mathbb{R}^3)$ a (unique) solution

$$(g,h) \in L_p((0,T);H^2_q(\Omega;\mathbb{M}^3 \times \mathbb{R}^3)) \cap H^1_p((0,T);L_q(\Omega;\mathbb{M}^3 \times \mathbb{R}^3))$$

for the system

$$\begin{cases} \partial_t G + \mathcal{G}_q G = f_2 & \text{in} \quad \Omega, \\ G = 0 & \text{on} \quad \partial\Omega, \\ \partial_t h + \mathcal{D}_q(m)h = f_3 & \text{in} \quad \Omega, \\ \partial_\nu h = 0 & \text{on} \quad \partial\Omega, \\ (G(0),h(0)) = (0,0). \end{cases}$$

Easy computations show that $\mathcal{B}_q(m)h \in L_p((0,T);L_q(\Omega;\mathbb{R}^3))$ for $m \in C^1(\overline{\Omega};\mathbb{R}^3)$. From the maximal L_p-regularity property of $\mathcal{A}_q$, we thus infer that there exists a (unique) vector $v \in L_p((0,T);H^2_{q,\sigma}(\Omega;\mathbb{R}^3)) \cap H^1_p((0,T);L_{q,\sigma}(\Omega;\mathbb{R}^3))$ that solves

$$\begin{cases} \partial_t v + \mathcal{A}_q v = -P_H \mathcal{B}_q(m)h + f_1 & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega, \\ v(0) = 0. \end{cases}$$

Hence (v,G,h) is the unique solution of (2.8). This shows that

$$A_q(m) \in \mathcal{MR}_p(X_1,X_0) \quad \text{for each } m \in C^1(\overline{\Omega};\mathbb{R}^3). \tag{2.9}$$

In addition, we define for $z=(u,F,m)$

$$G(z) := \left(P_H\left[\nabla\cdot(FF^{\mathsf{T}}) - u\cdot\nabla u\right], (\nabla u)^{\mathsf{T}}F - u\cdot\nabla F, \alpha|\nabla m|^2\, m - u\cdot\nabla m\right). \tag{2.10}$$

Given any $1<p,q<\infty$, $T>0$ and $\mu \in (1/p,1]$, we set

$$\mathbb{E}_{0,\mu}(T) := L_{p,\mu}((0,T);X_0) \quad \text{and} \quad \mathbb{E}_{1,\mu}(T) := L_{p,\mu}((0,T);X_1) \cap H^1_{p,\mu}((0,T);X_0).$$

It is well known that

$$\mathbb{E}_{1,\mu}(T) \hookrightarrow C([0,T];X_{\gamma,\mu}) \quad \text{where} \quad X_{\gamma,\mu} := (X_0,X_1)_{\mu-1/p,p}.$$

See [18], or [14, Theorem 3.4.8]. Observe that by [19, Theorem 3.4] and [20, Theorem 4.3.3], the triple $(u,F,m) \in B^{2\mu-2/p}_{qp}(\Omega;\mathbb{R}^{15})$ belongs to $X_{\gamma,\mu}$ iff

$$\begin{aligned} &u \in B^{2\mu-2/p}_{qp,\sigma}(\Omega;\mathbb{R}^3) && \text{and} \quad u=0 \text{ on } \partial\Omega, \\ &F \in B^{2\mu-2/p}_{qp}(\Omega;\mathbb{M}^3) && \text{and} \quad F=0 \text{ on } \partial\Omega, \\ &m \in B^{2\mu-2/p}_{qp}(\Omega;\mathbb{R}^3) && \text{and} \quad \partial_\nu m=0 \text{ on } \partial\Omega, \end{aligned} \tag{2.11}$$

where $B^{2\mu-2/p}_{qp,\sigma}(\Omega;\mathbb{R}^3) := B^{2\mu-2/p}_{qp}(\Omega;\mathbb{R}^3) \cap L_{q,\sigma}(\Omega;\mathbb{R}^3)$. In order for $\partial_\nu m$ to be defined, we assume that $2\mu-2/p-1/q>1$.

One readily verifies that

$$A_q \in C^\omega(X_{\gamma,\mu}, \mathcal{L}(X_1, X_0)), \quad G \in C^\omega(X_{\gamma,\mu}, X_0), \tag{2.12}$$

with ω being the notation for real analyticity, as long as

$$X_{\gamma,\mu} \hookrightarrow C^1(\overline{\Omega}; \mathbb{R}^{15}).$$

The above embedding holds whenever $\mu \in \left(\frac{1}{2} + \frac{1}{p} + \frac{3}{2q}, 1\right]$.

By the definitions (2.5), (2.7) and (2.10), and the relation (2.6), one sees that system (2.3) can be recast as the abstract evolutionary system

$$\partial_t z + A_q(m)z = G(z), \quad z(0) = z_0 = (u_0, F_0, m_0). \tag{2.13}$$

We have the following result on existence and uniqueness of solutions of (2.13).

Proposition 2.3. *Suppose $\mu \in \left(\frac{1}{2} + \frac{1}{p} + \frac{3}{2q}, 1\right]$ and let $z_0 \in X_{\gamma,\mu}$. Then there exists $T = T(z_0)$ such that the evolution equation (2.13) admits a unique solution $z \in \mathbb{E}_{1,\mu}(T)$. Each solution can be extended to a maximal existence interval $[0, T_+(z_0))$ in the sense that*

- *(i) either $T_+(z_0) = \infty$ or*
- *(ii) $\lim_{t\to T_+(z_0)} z(t)$ does not exist in $X_{\gamma,\mu}$.*

Moreover, z enjoys the additional regularity properties

$$z \in C([0, T_+); X_{\gamma,\mu}) \cap C^\omega((0, T_+); X_1) \cap C^\omega((0, T_+) \times \Omega; \mathbb{R}^{15}). \tag{2.14}$$

Proof. The existence, uniqueness and time regularity follow from (2.9), (2.12) and [14, Theorems 5.1.1 and 5.2.1 and Corollary 5.1.2], see also [21, Theorem 2.1]. The joint space–time regularity (2.14) can be proved by means of the parameter trick in [22], see also [14, Section 9.4.1]. □

Next, we show that solutions of (2.13) give rise to solutions of (2.3), and vice versa.

Proposition 2.4. *Let $T > 0$ be given. The following statements are equivalent:*

- *(a) (2.13) has a solution $(u, F, m) \in \mathbb{E}_{1,\mu}(T)$.*
- *(b) (2.3) has a solution $(u, F, m, \pi) \in \mathbb{E}_{1,\mu}(T) \times L_{p,\mu}((0, T); \dot{H}^1_q(\Omega))$.*

Proof. (a)⇒(b): Suppose $z = (u, F, m) \in \mathbb{E}_{1,\mu}(T)$ solves (2.13) on $[0, T]$. Let

$$v = \mu_s \Delta u - u \cdot \nabla u - \nabla \cdot (\nabla m \odot \nabla m) + \nabla \cdot (FF^\mathsf{T}).$$

Then $v \in L_{p,\mu}((0, T); L_q(\Omega; \mathbb{R}^3))$. For $t \in (0, T)$, let $\nabla\psi_{v(t)} \in L_q(\Omega; \mathbb{R}^3)$ be the unique solution of

$$(\nabla\psi_{v(t)}|\nabla\phi)_\Omega = (v(t)|\nabla\phi)_\Omega, \quad \forall\phi \in \dot{H}^1_{q'}(\Omega),$$

where $(\cdot|\cdot)_\Omega$ is the inner product of $L_2(\Omega; \mathbb{R}^3)$ and q' is the Hölder dual of q. Then $P_H v(t) = v(t) - \nabla\psi_{v(t)}$ by the definition of the Helmholtz projection. Let $\pi = \psi_v$. Then $\pi \in L_{p,\mu}((0, T); \dot{H}^1_q(\Omega))$, and noting that $P_H \partial_t u = \partial_t u$, we conclude that (u, F, m, π) is a solution of (2.3) in the regularity class $\mathbb{E}_{1,\mu}(T) \times L_{p,\mu}((0, T); \dot{H}^1_q(\Omega))$.

(b)⇒(a): Suppose $(u, F, m, \pi) \in \mathbb{E}_{1,\mu}(T) \times L_{p,\mu}((0, T); \dot{H}^1_q(\Omega))$ solves (2.3). Applying P_H to the equation governing u in (2.3), it is an easy task to check that $(u, F, m) \in \mathbb{E}_{1,\mu}(T)$ solves (2.13). □

We are now ready for our main result on existence and uniqueness of solutions for system (2.2), or equivalently, system (1.1).

Theorem 2.5. *Let $p, q \in (1, \infty)$ and $\mu \in \left(\frac{1}{2} + \frac{1}{p} + \frac{3}{2q}, 1\right]$. Suppose that*

$$z_0 = (u_0, F_0, m_0) \in B^{2\mu-2/p}_{qp,\sigma}(\Omega; \mathbb{R}^3) \times B^{2\mu-2/p}_{qp}(\Omega; \mathbb{M}^3) \times B^{2\mu-2/p}_{qp}(\Omega; \mathbb{R}^3)$$

satisfies the compatibility conditions $(u_0, F_0, \partial_\nu m_0) = 0$ on $\partial\Omega$. Then there exists a unique solution

$$(u, F, m, \pi) \in \left[H^1_{p,\mu}((0,T); X_0) \times L_{p,\mu}((0,T); X_1)\right] \times L_{p,\mu}((0,T); \dot{H}^1_q(\Omega))$$

of (2.3) for some $T = T(z_0) > 0$. Each solution can be extended to a maximal existence interval $[0, T_+(z_0))$. Moreover, $(z, \pi) = (u, F, m, \pi)$ enjoys the additional regularity

$$z \in C([0, T_+); X_{\gamma,\mu}) \cap C^\omega((0, T_+); X_1) \text{ and } (z, \pi) \in C^\omega((0, T_+) \times \Omega; \mathbb{R}^{16}). \tag{2.15}$$

If $|m_0| \equiv 1$, then the solution also satisfies

$$|m(t)| \equiv 1, \quad t \in [0, T_+(z_0)). \tag{2.16}$$

Proof. The assertions in the first part of the statement follow readily from Propositions 2.3 and 2.4. It then only remains to show that the condition $|m(t)| \equiv 1$ holds for every $t \in [0, T_+(z_0))$, provided $|m_0| \equiv 1$.

Suppose then that $|m_0| \equiv 1$. Let $T \in (0, T_+(z_0))$ be fixed and set $\varphi = |m|^2 - 1$. We note that

$$\begin{aligned} &m \in C^1((0,T); H^2_q(\Omega; \mathbb{R}^3)) \\ &\varphi \in C([0,T]; B^{2\mu-2/p}_{qp}(\Omega)) \cap C^1((0,T); H^2_q(\Omega)). \end{aligned}$$

Indeed, (2.15) implies that $m \in C([0,T]; B^{2\mu-2/p}_{qp}(\Omega; \mathbb{R}^3)) \cap C^1((0,T); H^2_q(\Omega; \mathbb{R}^3))$. The condition $\mu \in \left(\frac{1}{2} + \frac{1}{p} + \frac{3}{2q}, 1\right]$ guarantees that $B^{2\mu-2/p}_{qp}(\Omega)$ and $H^2_q(\Omega)$ are Banach algebras. The asserted regularity of φ thus holds. Taking the dot product of the equation

$$\partial_t m + u \cdot \nabla m = \alpha(\Delta m + |\nabla m|^2\, m) - \beta m \times \Delta m$$

with m and using the relations $\partial_t |m|^2 = 2\partial_t m \cdot m,\ \Delta |m|^2 = 2\Delta m \cdot m + 2|\nabla m|^2$ results in

$$\begin{cases} \partial_t \varphi + u \cdot \nabla \varphi - \alpha \Delta \varphi - 2\alpha |\nabla m|^2 \varphi = 0 & \text{in} \quad \Omega, \\ \partial_\nu \varphi = 0 & \text{on} \quad \partial\Omega, \\ \varphi(0) = 0. \end{cases} \tag{2.17}$$

Multiplying both sides of (2.17) with φ and integrating over Ω yields

$$\frac{d}{dt} \int_\Omega \frac{1}{2} \varphi^2 \, dx + \alpha \int_\Omega |\nabla \varphi|^2 \, dx = 2\alpha \int_\Omega |\nabla m|^2 \varphi^2 \, dx, \quad t \in (0,T).$$

As $m \in C([0,T]; B^{2\mu-2p}_{qp}(\Omega; \mathbb{R}^3)) \hookrightarrow C([0,T]; C^1(\overline{\Omega}; \mathbb{R}^3))$, we obtain the following integral inequality

$$\frac{d}{dt} \int_\Omega \varphi^2(t) \, dx \le 4\alpha \|\nabla m(t)\|^2_\infty \int_\Omega \varphi^2(t) \, dx, \quad t \in (0,T).$$

Applying the Gronwall inequality we get

$$\max_{0 \le t \le T} \int_\Omega \varphi^2(t) \, dx \le \exp\left(4\alpha \int_0^T \|\nabla m(t)\|^2_\infty \, dx\right) \int_\Omega \varphi^2(0) \, dx = 0.$$

This implies $\varphi \equiv 0$ in $Q_T = [0,T] \times \Omega$. In other words, $|m| \equiv 1$ in Q_T. As this is true for every $T \in (0, T_+(z_0))$ we obtain that $|m(t)| \equiv 1$ for any $t \in (0, T_+(z_0))$. As (1.1) and (2.2) are equivalent, we have proved the assertions of the theorem. □

Remark 2.6. Let p, q, μ be as in Theorem 2.5. The assertions of Theorem 2.5, with exception of the higher time regularity stated in (2.15), still hold if we pose the nonhomogeneous boundary conditions

$$(u, F, m) = (u_D, F_D, m_N) \quad \text{on } \partial\Omega, \tag{2.18}$$

where

$$\begin{aligned} u_D &\in F^{1-1/2q}_{pq,\mu}((0,T); L_q(\partial\Omega;\mathbb{R}^3)) \cap L_{p,\mu}((0,T); W^{2-1/q}_q(\partial\Omega;\mathbb{R}^3)) \\ F_D &\in F^{1-1/2q}_{pq,\mu}((0,T); L_q(\partial\Omega;\mathbb{M}^3)) \cap L_{p,\mu}((0,T); W^{2-1/q}_q(\partial\Omega;\mathbb{M}^3)) \\ m_N &\in F^{1/2-1/2q}_{pq,\mu}((0,T); L_q(\partial\Omega;\mathbb{R}^3)) \cap L_{p,\mu}((0,T); W^{1-1/q}_q(\partial\Omega;\mathbb{R}^3)) \end{aligned}$$

and the initial data satisfy the compatibility conditions

$$(u_D(0), F_D(0), m_N(0)) = (u_0, F_0, \partial_\nu m_0) \quad \text{on } \partial\Omega.$$

See [14, Theorem 6.3.2]. Here $F^s_{pq,\mu}$ are the Triebel–Lizorkin spaces with temporal weight.

3. Stability and asymptotic behavior

The last two sections are devoted to a discussion of the asymptotic behavior of solutions (u, F, m, π) to (1.1). In view of Proposition 2.4, the pressure π can be obtained from $z = (u, F, m)$. For this reason, it suffices to restrict our attention to a solution $z = (u, F, m)$ of (2.13).

The 3-dimensional subspace

$$\mathcal{E}_0 := \{0\} \times \{0_3\} \times \mathbb{R}^3 \ \text{ of } X_1$$

is clearly contained in the set $\mathcal{E}_1$ of equilibria of (2.13), where 0_3 is the 3×3 matrix with zero entries. We refer to Remark 4.3(a) for more information on $\mathcal{E}_1$.

At each $z_* = (0, 0_3, m_*) \in \mathcal{E}_0$, the linearization of (2.13) is given by

$$\partial_t z + A_* z = 0, \quad z(0) = z_0, \tag{3.1}$$

where $z = (u, F, m)$ and

$$A_* z = (-\mu_s P_H \Delta u, -\kappa\Delta F, (\beta \mathsf{M}(m_*) - \alpha I_3)\Delta m).$$

Since Ω is bounded, the spectrum of A_* consists only of eigenvalues. Suppose that $A_* z = \lambda z$ for some $\lambda \in \mathbb{C}$. By elliptic regularity theory, we can assume that $z \in H^2_q(\Omega;\mathbb{C}^{15})$ for $q \geq 2$. Taking the inner product of $A_* z = \lambda z$ with $\overline{z}$, where $\overline{z}$ denotes the complex conjugate of z, direct computations lead to

$$\operatorname{Re}\lambda \left(\|u\|_2^2 + \|F\|_2^2 + \|m\|_2^2\right) = \mu_s\|\nabla u\|_2^2 + \kappa\|\nabla F\|_2^2 + \alpha\|\nabla m\|_2^2,$$

which implies that $\operatorname{Re}\lambda \geq 0$. Here we have used the anti-symmetry of $\mathsf{M}(m_*)$ to conclude that

$$\operatorname{Re}(\mathsf{M}(m_*)\Delta m|\overline{m})_\Omega = 0.$$

Indeed, as the entries of $\mathsf{M}(m_*)$ are constant, we obtain

$$(\mathsf{M}(m_*)\Delta m|\overline{m})_\Omega = (\Delta(\mathsf{M}(m_*)m)|\overline{m})_\Omega = (\mathsf{M}(m_*)m|\overline{\Delta m})_\Omega,$$

where we set $z \cdot \overline{w} = z_j \overline{w_j}$ for $z, w \in \mathbb{C}^3$. The anti-symmetry of $\mathsf{M}(m_*)$ implies

$$(\mathsf{M}(m_*)\Delta m|\overline{m})_\Omega = -(\Delta m|\mathsf{M}(m_*)\overline{m})_\Omega = -\overline{(\mathsf{M}(m_*)m|\overline{\Delta m})}_\Omega.$$

This readily yields $\operatorname{Re}(\mathsf{M}(m_*)\Delta m|\overline{m})_\Omega = 0$. When $\operatorname{Re}\lambda = 0$, one concludes from the above that

$$\|\nabla u\|_2 = \|\nabla F\|_2 = \|\nabla m\|_2 = 0.$$

Combined with the boundary conditions, this shows that $z = (u, F, m) \in \mathcal{E}_0$. Further, we infer that $A_* z = 0$. Thus $\sigma(A_*) \cap i\mathbb{R} = \{0\}$ and $N(A_*) = \mathcal{E}_0$.

To show $\{0\}$ is a semi-simple eigenvalue, we will prove that $N(A_*) = N(A_*^2)$. Assume that $w = (v, f, h) \in N(A_*^2)$. Then there exists $z = (0, 0_3, m) \in N(A_*)$ such that $A_* w = z$. Then by the divergence theorem, the boundary condition $\partial_\nu h = 0$ and the fact that m as well as m_* are constant,

$$\|z\|_2^2 = (A_* w|z) = ((\beta \mathsf{M}(m_*) - \alpha I_3)\Delta h|m) = 0.$$

We conclude that $z = 0$ and thus $w \in N(A_*)$. This shows that $\{0\}$ is semi-simple. As $\mathcal{E}_0$ is a linear space, we clearly have $T_{z_*}\mathcal{E}_0 = N(A_*)$.

If follows from [23, Remark 2.2], see also [14, Remarks 5.3.2], that all equilibria close to z_* are contained in a manifold $\mathcal{M}$ of dimension $3 = \dim(N(A_*))$, where we used the fact that the center space X^c coincides with $N(A_*)$ as $\{0\}$ is semi-simple. Since the dimension of $\mathcal{E}_0$ is also 3, we conclude that there exists an open neighborhood $V_* \subset X_1$ of z_* such that $\mathcal{M} \cap V_* = \mathcal{E}_0 \cap V_*$. Hence, the neighborhood V_* contains no other equilibria than the elements of $\mathcal{E}_0$, that is, $V_* \cap \mathcal{E}_0 = V_* \cap \mathcal{E}_1$.

We have, thus, shown that z_* is normally stable, see [14, Theorem 5.3.1] for a definition.

Theorem 3.1. *Let $p, q \in (1, \infty)$ and $\mu \in \left(\frac{1}{2} + \frac{1}{p} + \frac{3}{2q}, 1\right]$.*

Then each equilibrium $z_* = (0, 0_3, m_*)$ with $m_* \in \mathbb{S}^2$ is stable in the topology of $X_{\gamma,\mu}$. There exists $\varepsilon > 0$ such that any solution (u, F, m, π) of (1.1) with initial value $z_0 = (u_0, F_0, m_0) \in X_{\gamma,\mu}$ satisfying $\|z_0 - z_*\|_{X_{\gamma,\mu}} \le \varepsilon$ exists globally and converges to some $z_\infty = (0, 0_3, m_\infty)$ with $m_\infty \in \mathbb{S}^2$ in the topology of $X_{\gamma,1}$ at an exponential rate as $t \to \infty$.

Proof. Given an equilibrium $z_* = (0, 0_3, m_*) \in \mathcal{E}_0$, we infer from [14, Theorem 5.3.1] and [24, Proposition 5.1] that there exists $\varepsilon > 0$ such that any solution $z = (u, F, m)$ of (2.13) with initial data $z_0 = (u_0, F_0, m_0) \in X_{\gamma,\mu}$ satisfying the conditions $|m_0| \equiv 1$ and $\|z_0 - z_*\|_{X_{\gamma,\mu}} \le \varepsilon$ exists globally and converges at an exponential rate to some $z_\infty = (0, 0_3, m_\infty)$ with $m_\infty =$ constant, in the topology of $X_{\gamma,1}$ as $t \to \infty$. By Proposition 2.4, we can determine a pressure π such that (z, π) solves (2.3) on $\mathbb{R}_+$. Furthermore, since $|m_0| \equiv 1$, we infer that $|m(t)| \equiv 1$ for all $t \ge 0$, which implies that (z, π) solves (1.1) on $\mathbb{R}_+$. Finally, we conclude that $m_\infty \in \mathbb{S}^2$. $\square$

4. Lyapunov functional and global solutions

Let

$$\mathsf{E} := \mathsf{E}(u, F, m) := \frac{1}{2}\int_\Omega \left(|u|^2 + |F|^2 + |\nabla m|^2\right) dx. \tag{4.1}$$

We show that the energy E is dissipated.

Proposition 4.1. *Let (u, F, m, π) be a solution of (1.1) with initial value z_0 satisfying the assertions of Theorem 2.5. Then*

$$\frac{d}{dt}\mathsf{E}(t) = -\int_\Omega \left(\mu_s|\nabla u(t,x)|^2 + \kappa|\nabla F(t,x)|^2 + \alpha|\Delta m(t,x) + |\nabla m(t,x)|^2 m(t,x)\,|^2\right) dx$$

for $t \in (0, T_+(z_0))$. Moreover, E is a strict Lyapunov functional for (1.1).

Proof. Let z_0 be an initial value satisfying the assumptions of Theorem 2.5. Then (1.1) admits a unique solution (u, F, m, π) in the regularity class stated in the Theorem. In particular, $z = (u, F, m)$ enjoys the regularity property

$$z \in C([0, T_+); X_{\gamma,\mu}) \cap C^1((0, T_+); X_1),$$

with $T_+ = T_+(z_0)$. In the following, we suppress the time variable $t \in (0, T_+)$. A straightforward computation, using the boundary condition $\partial_\nu m = 0$, yields

$$\frac{d}{dt}\mathsf{E} = \int_\Omega (\partial_t u \cdot u + \partial_t F : F - \partial_t m \cdot \Delta m)\, dx.$$

We have

$$\begin{aligned}\int_\Omega \partial_t u \cdot u\, dx &= \int_\Omega \left[\mu_s \Delta u - u \cdot \nabla u - \nabla \pi - \nabla \cdot (\nabla m \odot \nabla m) + \nabla \cdot (F F^\mathsf{T})\right] \cdot u\, dx \\ &= \int_\Omega \left(-\mu_s |\nabla u|^2 - (u \cdot \nabla m) \cdot \Delta m - F F^\mathsf{T} : \nabla u\right) dx,\end{aligned} \tag{4.2}$$

where we used $\nabla \cdot u = 0$, the boundary condition $u = 0$ on $\partial\Omega$, (1.7), and the relations

$$\nabla \cdot (\nabla m \odot \nabla m) = \nabla m\, \Delta m + \frac{1}{2}\nabla(|\nabla m|^2), \quad (\nabla m\, \Delta m) \cdot u = (u \cdot \nabla m) \cdot \Delta m.$$

Moreover,

$$\begin{aligned}\int_\Omega \partial_t F : F\, dx &= \int_\Omega \left[(\nabla u)^\mathsf{T} F - u \cdot \nabla F + \kappa \Delta F\right] : F\, dx \\ &= \int_\Omega \left(F F^\mathsf{T} : \nabla u - \kappa |\nabla F|^2\right) dx,\end{aligned} \tag{4.3}$$

where we employed the condition $\nabla \cdot u = 0$, the boundary condition $F = 0$ on $\partial\Omega$, and the relations

$$(\nabla u)^\mathsf{T} F : F = F F^\mathsf{T} : \nabla u, \quad 2(u \cdot \nabla F) : F = u \cdot \nabla |F|^2.$$

Observing that $(\partial_t m + u \cdot \nabla m) \cdot m = 0$, we obtain

$$\begin{aligned}\int_\Omega (\partial_t m + u \cdot \nabla m) \cdot \Delta m\, dx &= \int_\Omega (\partial_t m + u \cdot \nabla m) \cdot (\Delta m + |\nabla m|^2\, m)\, dx \\ &= \int_\Omega (\alpha(\Delta m + |\nabla m|^2\, m) - \beta\, m \times \Delta m) \cdot (\Delta m + |\nabla m|^2\, m)\, dx \\ &= \alpha \int_\Omega |\Delta m + |\nabla m|^2\, m|^2\, dx\end{aligned} \tag{4.4}$$

as $m \times \Delta m$ is perpendicular to both m and Δm. Combining the results in (4.2)–(4.4) readily yields the assertion.

Hence, E is non-increasing along solutions and, thus, is a Lyapunov functional. If, for any time $t \in I := (t_1, t_2) \subset (0, T_+(z_0))$ with some $0 \le t_1 < t_2$, $\frac{d}{dt}\mathsf{E}(t) = 0$, then

$$\|\nabla u(t)\|_2 = \|\nabla F(t)\|_2 = \|\Delta m(t) + |\nabla m(t)|^2 m(t)\|_2 = 0.$$

Combining with the boundary conditions, we infer that

$$u(t) = 0, \quad F(t) = 0_3, \quad t \in I.$$

This readily yields $(\partial_t u(t), \partial_t F(t)) = (0, 0_3)$ for all $t \in I$. Moreover, the condition $\|\Delta m(t) + |\nabla m(t)|^2 m(t)\|_2 = 0$ implies that

$$\Delta m(t) + |\nabla m(t)|^2 m(t) = 0 \quad \text{in } \Omega, \quad t \in I. \tag{4.5}$$

Taking the cross product of both sides of (4.5) by $m(t)$ yields

$$m(t) \times \Delta m(t) = -|\nabla m(t)|^2 m(t) \times m(t) = 0 \quad \text{in } \Omega.$$

Therefore, $\partial_t m(t) = 0$ for all $t \in I$. Hence, $(\partial_t u(t), \partial_t F(t), \partial_t m(t)) = (0, 0_3, 0)$ for all $t \in I$, and this means that the system is at equilibrium for $t \in I$. To sum up, we have proved that $\mathsf{E} : X_{\gamma,\mu} \to \mathbb{R}$ is a strict Lyapunov functional for (1.1). □

The arguments above additionally yield a characterization of the set of equilibria of (1.1).

Corollary 4.2. *The set of equilibria of (1.1) is given by*

$$\mathcal{E} = \{(0, 0_3, m_*, \pi_*)\}, \tag{4.6}$$

where $m_* \in H^2_q(\Omega)$ solves the constrained nonlinear elliptic problem

$$\begin{cases} \Delta m_* + |\nabla m_*|^2 m_* = 0 & \text{in} \quad \Omega, \\ |m_*| \equiv 1 & \text{in} \quad \Omega, \\ \partial_\nu m_* = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4.7}$$

and $\pi_* = -\frac{1}{2}|\nabla m_*|^2 + C$ for some constant C.

Proof. We have already shown in the proof of Proposition 4.1 that any equilibrium of (1.1) is given by $(0, 0_3, m_*)$, where m_* solves (4.7). Hence, at equilibrium, we are left with the relation $\nabla \pi_* = -\nabla \cdot (\nabla m_* \otimes \nabla m_*)$. We have

$$\begin{aligned} [-\nabla \cdot (\nabla m_* \otimes \nabla m_*)]_i &= -\partial_i m_* \cdot \Delta m_* - (\partial_i \partial_j m_*) \cdot \partial_j m_* \\ &= \partial_i m_* \cdot m_* |\nabla m_*|^2 - \frac{1}{2}\partial_i |\nabla m_*|^2 = -\frac{1}{2}\partial_i |\nabla m_*|^2, \end{aligned}$$

where we used the relations $\Delta m_* = -|\nabla m_*|^2 m_*$ and $\partial_i m_* \cdot m_* = \frac{1}{2}\partial_i |m_*|^2 = 0$. Hence, $\nabla \pi_* = -\frac{1}{2}\nabla|\nabla m_*|^2$ and the assertion for π_* follows. □

Remark 4.3. (a) We note that for solutions of system (2.3), that is, in case the condition $|m| \equiv 1$ is dropped in (2.2), we can only conclude that

$$\frac{d}{dt}\mathsf{E} = -\int_\Omega \left(\mu_s |\nabla u|^2 + \kappa |\nabla F|^2 + \alpha(|\Delta m|^2 + |\nabla m|^2 (m \cdot \Delta m))\right) dx.$$

As the term $m \cdot \Delta m$ does not have a sign, we can no longer derive the characterization (4.6) for the set of equilibria, $\mathcal{E}_1$, of (2.3), respectively (2.13). However, as shown in Section 3, we can conclude that for every $z_* \in \mathcal{E}_0$ there exists a neighborhood V_* in X_1 such that $\mathcal{E}_1 \cap V_* = \mathcal{E}_0 \cap V_*$.

(b) It is claimed in [25, Lemma 5.2], see also Prüss and Simonett [14, Lemma 12.2.4], that the nonlinear problem (4.7) admits only constant solutions $m_* \in \mathbb{S}^2$. However, this assertion is not correct in the form stated, as the following example shows: Let $\Omega = \{x \in \mathbb{R}^3 : 0 < r_1 < |x| < r_2\}$ and $m_* : \Omega \to \mathbb{S}^2$ be defined by $m_*(x) = x/|x|$. Then m_* is a (non-constant) solution of (4.7).

Theorem 4.4. *Let p, q, μ, z_0 and $T_+(z_0)$ be as in Theorem 2.5. Suppose that the solution (u, F, m, π) of (1.1) satisfies*

$$z = (u, F, m) \in BC([\delta, T_+(z_0)); X_{\gamma,\bar{\mu}})$$

for some $\delta \in (0, T_+(z_0))$ and $\bar{\mu} \in (\mu, 1]$. Then z exists globally and $\operatorname{dist}(u(t), \mathcal{E}) \to 0$ in $X_{\gamma,1}$ as $t \to \infty$, where $\mathcal{E}$ is the set of equilibria of (1.1).

Proof. Given any initial value z_0, we define the ω-limit set of (2.13) as

$$\omega(z_0) := \{w \in X_{\gamma,\mu} : \exists t_n \to \infty \text{ s.t. } \|z(t_n) - w\|_{X_{\gamma,1}} = 0 \text{ as } n \to \infty\}.$$

[14, Theorem 5.7.1] implies that $z(\cdot)$ exists globally and the orbit $\{z(t)\}_{t \geq \delta}$ is relatively compact in $X_{\gamma,1}$. By [14, Theorem 5.7.2], $\omega(z_0)$ is nonempty, compact and $\omega(z_0) \subset \mathcal{E}$. Further, we can infer that $\operatorname{dist}(z(t), \mathcal{E}) \to 0$ in $X_{\gamma,1}$ as $t \to \infty$. □

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