Research Article

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Compactness of *M*-uniform domains and optimal thermal insulation problems

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Abstract: In this paper, we will consider an optimal shape problem of heat insulation introduced by [D. Bucur, G. Buttazzo and C. Nitsch, Two optimization problems in thermal insulation, *Notices Amer. Math. Soc.* **64** (2017), no. 8, 830–835]. We will establish the existence of optimal shapes in the class of M-uniform domains. We will also show that balls are stable solutions of the optimal heat insulation problem.

Keywords: M-uniform domains, optimal heat insulation problem, second-order variations

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1 Introduction

1.1 Background

In this paper, motivated by Bucur–Buttazzo–Nitsch in their papers [10, 11], we consider the thermal insulation problem of designing the optimal shape Ω of $\mathbb{R}^n$ which represents a thermally conducting body, and determining the best distribution of a given amount of insulating material around Ω; the thickness of the insulating material is assumed to be very small with respect to the size of Ω, so the material density is assumed to be a nonnegative function defined on the boundary $\partial\Omega$. A rigorous approach is to consider a limit problem when the thickness of the insulating layer goes to zero and simultaneously the conductivity in the layer goes to zero.

Mathematically, this amounts to consider the limit of the family of functionals, as $\varepsilon \to 0$,

$$F_\varepsilon(u, h, \Omega) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\varepsilon}{2}\int_{\Sigma_\varepsilon} |\nabla u|^2\,dx - \int_\Omega fu\,dx, \tag{1.1}$$

over $u \in H_0^1(\Omega_\varepsilon)$, where $\Omega_\varepsilon = \Omega \cup \Sigma_\varepsilon$. Here Ω has a prescribed volume V_0,

$$\Sigma_\varepsilon = \{\sigma + t\nu(\sigma) : \sigma \in \partial\Omega,\ 0 \le t \le \varepsilon h(\sigma)\}$$

is the thin layer of thickness $\varepsilon h(\sigma)$ around $\partial\Omega$, and $h \in \mathscr{H}_m$, where

$$\mathscr{H}_m = \Big\{h \colon \partial\Omega \to \mathbb{R} \text{ is measurable},\ h \ge 0,\ \int_{\partial\Omega} h\,d\sigma = m\Big\}$$

and h denotes the distribution function of insulation material with fixed total amount $m > 0$.

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As in [1, 10], in the framework of Γ-convergence passing to the limit $\varepsilon \to 0$ in (1.1), we obtain the limit energy functional

$$\mathcal{F}_m(u,h,\Omega) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_{\partial\Omega} \frac{u^2}{h}\,d\sigma - \int_\Omega fu\,dx. \tag{1.2}$$

By [9], for any fixed u and Ω, if we minimize $F(u,h,\Omega)$ over $h \in \mathscr{H}_m$, then $F(u,h,\Omega)$ achieves its minimum when

$$h = m\frac{|u|}{\int_{\partial\Omega}|u|\,d\sigma}. \tag{1.3}$$

After substituting (1.3) for h into (1.2), we seek to minimize

$$\mathcal{J}_m(u,\Omega) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2m}\Big(\int_{\partial\Omega}|u|\,d\mathscr{H}^{n-1}\Big)^2 - \int_\Omega fu\,dx \tag{1.4}$$

over all $u \in H^1(\Omega)$, subject to the volume constraint $|\Omega| = V_0$. See [7] for earlier related work.

It was proved in [10] that, for every $f \in L^2(\Omega)$, if Ω is fixed, then the minimization of (1.4) admits a unique solution $u_\Omega \in H^1(\Omega)$, and moreover, if $\Omega = B_R$ and $f \equiv 1$, then

$$u_{B_R}(x) = \frac{R^2 - |x|^2}{2n} + \frac{m}{n^2\omega_n R^{n-2}},$$

where ω_n is the volume of unit ball in $\mathbb{R}^n$ and B_R is the ball of radius R centered at origin.

Stationary solutions were also obtained in [10]. More precisely, for a given smooth vector field $\eta \in C_0^\infty(\mathbb{R}^n)$ with $\int_\Omega \operatorname{div}\eta\,dx = 0$, let $F_t(x) := F(t,x)$ be the *flow map* generated by the vector field η, i.e. F_t solves the following ODE in $\mathbb{R}^n$:

$$\begin{cases} \dfrac{d}{dt}F(t,x) = \eta(F(t,x)), \\ F_0(x) = x. \end{cases}$$

It was proved in [10] that, for $f \equiv 1$, B_R is a stationary shape in the sense that

$$\frac{d}{dt}\Big|_{t=0}\mathcal{J}_m(u_t,\Omega_t) = 0,$$

where $u_t = u \circ F_t^{-1}$, $\Omega_t = F_t(B_R)$ and $|B_R| = V_0$.

Two open questions are asked by Bucur–Buttazzo–Nitsch in [11].

Problem 1.1. *Do the optimal shapes minimizing the energy functional* (1.4) *exist?*

Problem 1.2. *Is it true that B_R is a unique optimal shape when $f \equiv 1$?*

1.2 Existence of minimizers over convex domains

There has been a developed scheme for the existence of a minimizer to problem (1.4) over convex domains contained within a container B_R and H^1 function associated to such domains, due to the compactness properties of such domains; see [9, 20] and the survey book [19]. See also the paper [26] by Lin–Poon. Indeed, the existence of problem (1.4) relies on the following properties for convex domains: If $\Omega \subset B_R$ is convex, $|\Omega| = V_0 > 0$ and $u \in H^1(\Omega)$, then the following statements hold.

(1) (Uniform Poincaré inequality) There exists a universal constant $C > 0$, independent of (u,Ω), such that

$$\int_\Omega u^2\,dx \le C\Big(\int_\Omega |\nabla u|^2\,dx + \Big(\int_{\partial\Omega}|u|\,d\mathscr{H}^{n-1}(x)\Big)^2\Big). \tag{1.5}$$

This guarantees the uniform H^1-bound of u_i for any minimizing sequence (u_i,Ω_i) of $\mathcal{J}_m$.

(2) (Uniform Sobolev extension property) There exists a universal constant $C > 0$ independent of Ω such that, for each $u \in H^1(\Omega)$, there exists $\tilde{u} \in H^1(\mathbb{R}^n)$ such that $\tilde{u} = u$ in Ω, and

$$\|\tilde{u}\|_{H^1(\mathbb{R}^n)} \leq C\|u\|_{H^1(\Omega)}. \tag{1.6}$$

(3) (Compactness of convex domains) If Ω_i is a sequence of convex sets in B_R with $|\Omega_i| = V_0$, then there is a convex domain Ω such that $\Omega_i \to \Omega$ in L^1, and

$$\mathcal{H}^{n-1}\llcorner_{\partial\Omega_i} \to \mathcal{H}^{n-1}\llcorner_{\partial\Omega}$$

as convergence of Radon measures. See [3, 4] as well.

(4) (Lower semicontinuity of energy) From (1.5), (1.6) and the compactness of convex domains in B_R, for any minimizing sequence of pairs (u_i, Ω_i) to (1.4), there are Ω and $u \in H^1(\Omega)$ such that, up to a subsequence, $\Omega_i \to \Omega$ in L^1,

$$\begin{aligned} \int_\Omega |\nabla u|^2\,dx &\leq \liminf_{i\to\infty} \int_{\Omega_i} |\nabla u_i|^2\,dx, \\ \int_{\partial\Omega} |u|\,d\mathcal{H}^{n-1} &= \liminf_{i\to\infty} \int_{\partial\Omega_i} |u_i|\,d\mathcal{H}^{n-1} \end{aligned} \tag{1.7}$$

and

$$\lim_{i\to\infty} \int_{\Omega_i} f u_i\,dx = \int_\Omega f u\,dx.$$

The proof of (1.7) relies on the parametrization of $\partial\Omega$ by the sphere (see also [26]).

It is challenging to generalize this scheme for convex domains to more rough domains. In this context, we formulate the problem for a class of specified rough domains as follows.

1.3 Formulation of problem (1.4) over rough domains

We would like to study minimization problem (1.4) over some controllable rough domains, belonging to the class of Sobolev extension domains, with fixed volume. A natural class of Sobolev extension domains is the so-called M-uniform domain. In fact, when $n = 2$, M-uniforms domain are equivalent to extension domains for H^1 functions; see [24, 33]. Recall the following definition of M-uniform domain, which was first introduced in [16, 24].

Definition 1.1. For $M > 1$, a domain $\Omega \subset \mathbb{R}^n$ is called an M-uniform domain if, for any $x_1, x_2 \in \overline{\Omega}$, there is a rectifiable curve $\gamma\colon [0, 1] \to \overline{\Omega}$ such that $\gamma(0) = x_1$, $\gamma(1) = x_2$, and
(i) $\mathcal{H}^1(\gamma) \leq M|x_1 - x_2|$,
(ii) $d(\gamma(t), \partial\Omega) \geq \frac{1}{M}\min\{|\gamma(t) - x_1|, |\gamma(t) - x_2|\}$ for all $t \in [0, 1]$.

Roughly speaking, an M-uniform domain has no interior or exterior cusps, and it does not have very thin connections. The class of M-uniform domains contains convex domains in a ball, uniform Lipschitz domains and minimally smooth domain introduced in [32], and it can have a purely unrectifiable boundary, such as the complement of 4-corner Cantor set. This class has a wide range of sets.

We remark that if $\Omega \subset B_R$ is an M-uniform domain and $u \in H^1(\Omega)$, then u has an extension $\tilde{u}$ which is a BV function in an open neighborhood of B_R. Thus if Ω also has finite perimeter, then the trace of u can be defined on the reduced boundary $\partial^*\Omega$ in the sense that there exists a measurable function u^* on $\partial^*\Omega$ such that

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B_r(x)\cap\Omega} |u - u^*(x)|\,dy = 0, \quad \mathcal{H}^{n-1} \text{ a.e. } x \in \partial^*\Omega. \tag{1.8}$$

We call u^* the (interior) trace of u on $\partial^*\Omega$. The reader can refer to the monograph [2, Theorem 3.77].

Therefore, in the following, we formulate minimization problem (1.4) over rough sets as the minimization of

$$\mathcal{J}_m(u,\Omega) := \frac{1}{2}\int_{\Omega} |\nabla u|^2\,dx + \frac{1}{2m}\Big(\int_{\partial^*\Omega} |u^*|\,d\mathscr{H}^{n-1}\Big)^2 - \int_{\Omega} fu\,dx \tag{1.9}$$

over all $u \in H^1(\Omega)$, $|\Omega| = V_0$. We will prove that there is a minimizer to (1.9) among all sets of M-uniform domains with uniformly bounded perimeters, and thus we are able to solve problem (1.1) within this class of rough domains. The M-uniform condition of Ω plays an important role in generalizing the scheme for convex domains as mentioned above.

1.4 Main results

We will first state a theorem asserting the compactness of M-uniform domains in B_R, which does not require the domains to have finite perimeters.

Theorem 1.2. *For $M > 0$, let $\{\Omega_i\}$ be a sequence of M-uniform domains in B_R such that*

$$\inf_i \operatorname{diam}(\Omega_i) > 0. \tag{1.10}$$

Then there exists an M-uniform domain Ω such that, after passing to a subsequence, $\Omega_i \to \Omega$ in L^1, as $i \to \infty$.

Remark 1.3. Assumption (1.10) automatically holds if $|\Omega| = V_0 > 0$, i.e. there is $c = c(V_0, n) > 0$ such that $\operatorname{diam}(\Omega) \geq c > 0$.

With the help of Theorem 1.2, we can prove two uniform Poincaré inequalities for M-uniform domains; see Theorems 4.1 and 4.2 below. Applying Theorems 1.2 and 4.2, we can prove the following result.

Theorem 1.4. *For any $M > 0$, $\Lambda > 0$, $R > 0$ and $f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$,*

$$\mathcal{J}_m(u,\Omega) := \frac{1}{2}\int_{\Omega} |\nabla u|^2\,dx + \frac{1}{2m}\Big(\int_{\partial^*\Omega} |u^*|\,d\mathscr{H}^{n-1}\Big)^2 - \int_{\Omega} fu\,dx. \tag{1.11}$$

Then $\mathcal{J}_m$ admits a minimizer over

$$\mathcal{A} = \{(u,\Omega) : u \in H^1(\Omega),\ \Omega \text{ is an } M\text{-uniform domain in } B_R,\ |\Omega| = V_0 > 0,\ P(\Omega) \leq \Lambda\},$$

where $P(\Omega)$ is the perimeter of Ω.

We remark that, on the one hand, the M-uniformity assumption in Theorem 1.4 seems to be a natural sufficient condition for the existence of minimizers of $\mathcal{J}_m(u,\Omega)$, among $(u,\Omega) \in \mathcal{A}$, since it guarantees certain uniform Sobolev extension properties and a boundary Poincaré inequality that further control the L^1-norms of boundary traces u^* on the reduced boundary $\partial^*\Omega$ for minimizing sequences. On the other hand, it seems plausible that there may exist a minimizer for $\mathcal{J}_m(u,\Omega)$ in a more general class of domains; moreover, such a minimizer Ω may enjoy better regularity (e.g. a uniform domain with finite perimeter). We plan to further exploit this equation in the near future. Here we would like to mention a relevant work by Bucur [8].

It turns out that (1.11) can also be defined over the space of functions of special bounded variations (or SBV). Let $D \subset \mathbb{R}^n$ be a bounded smooth domain, and $f \in L^n(D)$, $f \geq 0$. Consider the following minimization problem:

$$\inf\Big\{\mathcal{J}(u) := \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2\,dx + \frac{1}{2m}\Big(\int_{J_u} (|u^+| + |u^-|)\,d\mathscr{H}^{n-1}\Big)^2 - \int_{\mathbb{R}^n} fu\,dx\Big\} \tag{1.12}$$

over $\mathcal{S} = \{u \in \mathrm{SBV}(\mathbb{R}^n, \mathbb{R}_+) : |\{u > 0\}| = V_0,\ |\operatorname{supp} u \setminus D| = 0,\ \mathscr{H}^{n-1}(J_u \cap \partial D) = 0\}$. Here ∇u is the absolutely continuous part of the distributional derivative Du with respect to the Lebesgue measure, and u^+ and u^- are one-sided limits of u on the jump set J_u of u. See [2] for the definition of $\mathrm{SBV}(\mathbb{R}^n)$. See [14, 17] for more background.

In this context, we are able to prove another existence result.

Theorem 1.5. *$\mathcal{J}(\cdot)$ admits a minimizer $u \in \mathcal{S}$.*

Remark 1.6. If $\Omega \subset D$ is an M-uniform domain of finite perimeter and $u \in H^1(\Omega)$ is a minimizer of problem (1.4), then $u\chi_\Omega \in \mathcal{S}$. On the other hand, for a minimizer v of (1.12), if $\Omega := \{x \in D : v(x) > 0\}$ is a subdomain of D, and v has no jump in Ω, i.e. $\mathscr{H}^{n-1}(J_v \cap \Omega) = 0$, where J_v is the jump set of v, then $v \in H^1(\Omega)$ and $(v\lfloor_\Omega, \Omega)$ is a minimizing pair of problem (1.9).

We will also study problem (1.2). This problem is extremely challenging. It seems to be open, among all C^2 domains, if $f \equiv 1$, whether a ball is an optimal configuration, let alone the uniqueness of an optimal shape. To see some of the difficulties to validate this conjecture, one may compare the functional $\mathcal{J}_m(u, \Omega)$ with the recently well studied energy functional

$$\tilde{J}(u, \Omega) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \beta \int_{\partial\Omega} u^2\,d\sigma - \int_\Omega u\,dx,$$

where β is a positive constant. Due to the linear splitting property of the regular functional $\tilde{J}$, a Steiner symmetrization argument can be implemented to show that *if (u, Ω), where $\Omega \subset \mathbb{R}^n$ is a smooth domain and $u \in H^1(\Omega)$, minimizes $\tilde{J}(v, U)$ among all $v \in H^1(U)$ and smooth domains $U \subset \mathbb{R}^n$ subject to the volume constraint $|U| = 1$, then Ω must be a ball of volume* 1. This can be done by Steiner symmetrization and analysis of an ODE with Robin boundary condition. We refer the interested readers to Bucur–Giacomini [13, page 9] for the detailed explanation. In contrast, it seems that none of the known symmetrization methods is applicable to the uniqueness of the minimization problem of $\mathcal{J}_m(u, \Omega)$ as described above.

In this paper, we manage to make some partial progress of problem (1.2). Our idea is to study this optimization problem through the method of domain variations. After some delicate calculations, which involves geometric evolution equations and an eigenvalue estimate of the Stekloff problem, we prove the following theorem.

Theorem 1.7. *For any $m > 0$, $R > 0$ and any smooth vector field $\eta \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$, with $\eta(x) \perp T_x\partial B_R$ for $x \in \partial B_R$, if the flow map F_t, associated with η, preserves the volume of B_R, then (u_R, B_R) is a stable, critical point of $\mathcal{J}_m(\cdot, \cdot)$ in the following sense:*

$$\frac{d}{dt}\Big|_{t=0} \mathcal{J}_m(u_{F_t(B_R)}, F_t(B_R)) = 0, \quad \frac{d^2}{dt^2}\Big|_{t=0} \mathcal{J}_m(u_{F_t(B_R)}, F_t(B_R)) \geq 0.$$

Here $u_{F_t(B_R)}$ is the unique minimizer of $\mathcal{J}_m(\cdot, F_t(B_R))$ in $H^1(F_t(B_R))$.

A couple of remarks related to Theorem 1.7 are in order.

Remark 1.8. We have learned from the referee that problem (1.2) has recently been solved by Pietra–Nitsch–Scala–Trombetti [30] through completely different arguments. However, we think that the shape derivative calculations made in Theorem 1.7 have their own interest and may have applications in different problems. In fact, in a very recent preprint [21], the second author and his coauthors have extended the second variation formula for general radial heat source functions $f(x) = f(|x|)$ along arbitrary directions and proved several interesting results on the stability and instability of $\mathcal{J}_m(u, \Omega)$ given by (1.11).

1.5 Some further remarks

The compactness of M-uniform domains with uniformly bounded perimeter was previously proved by Li–Wang [25], where the authors consider the minimization problem arising from the liquid crystal droplet problem:

$$J(u, \Omega) := \int_\Omega |\nabla u|^2\,dx + P(\Omega), \tag{1.13}$$

where $u \in H^1(\Omega, S^2)$ and $|\Omega| = V_0 > 0$. If (u_i, Ω_i) is a minimizing sequence to (1.13), then Ω_i automatically have uniformly bounded perimeters and thus have an L^1 limit up to a subsequence. It was proven in [25] that the limit is $\mathcal{L}^n$-equivalent to an M-uniform domain.

Motivated by a volume estimate result in [23] for general porous domains, we will show that M-uniform domains turn out to have uniformly bounded nonlocal perimeters and thus have an L^1 limit up to a subsequence by the fractional Sobolev compact embedding theorem; see Corollary 3.3. This together with the argument in [25] yields Theorem 1.2. Hence one may also consider problem (1.9) over M-uniform domains of finite perimeters, without additionally requiring that the perimeters are uniformly bounded as assumed in Theorem 1.4. The difficulty, however, is that, even if there is a limit and the limit of the domains in the minimizing sequence is still an M-uniform domain, it might not have finite perimeter, and thus the boundary integral term in (1.9) may not be well-defined. It would be very interesting to prove that the minimizing sequence of (1.9) does have uniformly bounded perimeters, instead of adding this as an assumption.

A byproduct of the compactness of M-uniform domains is a uniform Poincaré inequality for such domains; see Theorem 4.1. In [5], such a uniform Poincaré inequality was only proved for uniformly Lipschitz domains. Hence Theorem 4.1 generalizes this result of [5].

1.6 Notation

Throughout this paper, we adopt the standard notation. For a set $A \subset \mathbb{R}^n$, we let $A^r := \{x \in \mathbb{R}^n : d(x, A) < r\}$ and $A_r = \{x \in \mathbb{R}^n : B_r(x) \subset A\}$. Denote by $\mathscr{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure, and $d_H(\cdot,\cdot)$ denotes the Hausdorff distance between two sets. Denote by $\mathcal{L}^n$ the Lebesgue measure in $\mathbb{R}^n$. Let $|A|$ denote the Lebesgue measure of A. We also let $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$. We let $\partial^* A$ denote the reduced boundary of A. We use $\operatorname{diam}(A)$ to denote the diameter of A. Also, we always let ω_n be the volume of the unit ball in $\mathbb{R}^n$.

We let $\mathcal{M}_R$ be the class of all M-uniform domains contained in B_R, and we let $\mathcal{M}_{R,c}$ be the subclass of $\mathcal{M}_R$ such that any domain in the subclass has diameter bigger than or equal to $c > 0$. We always use u^* to denote the trace of u in the sense of (1.8). Last, when we say a set is a domain, we mean the set is a connected open set.

2 Preliminaries on rough domains

We start with some definitions.

Definition 2.1. For $c > 0$, $\mathcal{D}_c$ is the class of sets E satisfying

$$|B_r(x) \cap E| > cr^n$$

for any $x \in \partial E$ and $0 < r < \operatorname{diam}(E)$.

The next remark says that any set in $\mathcal{D}_c$ is $\mathcal{L}^n$-equivalent to its closure.

Remark 2.2. If $E \in \mathcal{D}_c$, then $E = \overline{E} \pmod{\mathcal{L}^n}$.

Proof. By the Lebesgue density theorem, if $E \in \mathcal{D}_c$, then $\partial E \subset E \pmod{\mathcal{L}^n}$. Hence $|\overline{E} \setminus E| = 0$. □

Remark 2.3. If $E \in \mathcal{D}_c$, then, for any $x \in \overline{E}$ and $0 < r < 2\operatorname{diam}(E)$, there is $c' = c'(c, n) > 0$ such that

$$|B_r(x) \cap E| \geq c' r^n.$$

Proof. There are two cases.
(a) If $r \geq 2d(x, \partial E)$, then there is $z \in \partial E$ and $B_{\frac{r}{2}}(z) \subset B_r(x)$; hence $|B_r(x) \cap E| \geq |B_{\frac{r}{2}}(z) \cap E| \geq c(\frac{r}{2})^n = 2^{-n} c r^n$.
(b) If $r \leq 2d(x, \partial E)$, then $B_{\frac{r}{2}}(x) \subset E$. Thus $|B_r(x) \cap E| \geq \omega_n(\frac{r}{2})^n$.
Hence there is $c' = c'(c, n) > 0$ such that $|B_r(x) \cap E| \geq c' \varepsilon^n$. □

The next proposition says M-uniform domains belong to the class $\mathcal{D}_c$.

Proposition 2.4. *If Ω is an M-uniform domain, with $\operatorname{diam}(\Omega) \geq c_0 > 0$, then $\Omega \in \mathcal{D}_c$ for some $c > 0$ depending only on M, n and $\frac{\operatorname{diam}(E)}{c_0}$.*

Proof. For any $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\Omega)$, we claim that there is a constant $c_1 = c_1(M) > 0$ such that there is a ball of radius $c_1 r$ contained in $B_r(x) \cap \Omega$. Indeed, since $0 < r < \operatorname{diam}(\Omega)$, there is $y \in \Omega \setminus B_{\frac{r}{2}}(x)$. Let $\gamma \subset \overline{\Omega}$ be the curve connecting x and y as in the definition of M-uniform domain. Choose $z \in \partial B_{\frac{1}{3}r}(x) \cap \gamma$. Then $z \in \Omega$ and $d(z, \partial\Omega) \geq \frac{1}{6M} r$. Hence if we choose $c_1(M) = \frac{1}{6M}$, then $B_{c_1(M)r}(z) \subset B_r(x) \cap \Omega$. In particular, for any $x \in \partial\Omega$ and any $0 < r < \operatorname{diam}\Omega$, $|B_r(x) \cap \Omega| \geq |B_{c_1(M)r}(z)| \geq c_1(M) r^n$. □

The following remark will be used in the proof of compactness of M-uniform domains.

Remark 2.5. If Ω is an M-uniform domain with $|\Omega| \geq c_0$, then there is $r_0 > 0$ depending only on M, n, c_0 such that Ω contains a ball of radius r_0.

Proof. By the isodiametric inequality, there is $c_1 = c_1(n) > 0$ such that $\operatorname{diam}(\Omega) > c_1$. From the proof of Proposition 2.4, Ω contains a ball of radius $\frac{1}{6M} c_1$. □

Similarly, we define $\mathcal{D}^c$ as follows.

Definition 2.6. For $c > 0$, let $\mathcal{D}^c$ be the class of sets E such that $|B_r(x) \cap E^c| > cr^n$ holds for any $x \in \partial E$ and $0 < r < \operatorname{diam}(E)$.

The following proposition is from [27, Proposition 12.19]. It says that, for any set $E \subset \mathbb{R}^n$, we can find an $\mathcal{L}^n$-equivalent set $\tilde{E}$ with a slightly better topological boundary such that $\partial\tilde{E} = \operatorname{spt}\mu_E$, where μ_E is the distributional perimeter measure of E.

Proposition 2.7. *For any Borel set $E \subset \mathbb{R}^n$, there exists an $\mathcal{L}^n$-equivalent set $\tilde{E}$ such that $|E \Delta \tilde{E}| = 0$ and for any $x \in \partial\tilde{E}$ and any $r > 0$, $0 < |\tilde{E} \cap B_r(x)| < \omega_n r^n$. In other words, $\operatorname{spt}\mu_E = \operatorname{spt}\mu_{\tilde{E}} = \partial\tilde{E}$.*

The next lemma concerns the L^1-convergence of sets in $\mathcal{D}_c$.

Lemma 2.8. *Suppose $D_i \subset B_{R_0}$ is a sequence of sets in $\mathcal{D}_c$ such that $D_i \to D$ in L^1. If we identify D with its $\mathcal{L}^n$-equivalent set $\tilde{D}$ as in Proposition 2.7, then $D \in \mathcal{D}_c$. Moreover, for any $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that, for $i > N$, the following properties hold.*
(i) $D \subset D_i^\varepsilon$.
(ii) $(D_i)_\varepsilon \subset D$.
(iii) $D_i \subset D^\varepsilon$.
In particular, D_i converges to D in the Hausdorff distance, i.e. $d_H(D_i, D) \to 0$ as $i \to \infty$.

Proof. We argue by contradiction. If (i) were false, then there would exist $x \in D$ such that $B_\varepsilon(x) \cap D_i = \emptyset$ for i sufficiently large. Hence, by the hypothesis and Proposition 2.7, we obtain $0 = |B_\varepsilon(x) \cap D_i| \to |B_\varepsilon(x) \cap D| > 0$, a contradiction.

If (ii) were false, there would be a sequence $x_i \in (D_i)_\varepsilon \setminus D$. We may assume $x_i \to x_0$. Thus $x_0 \in \partial D \cup D^c$. By Proposition 2.7, we have $\omega_n \varepsilon^n > |B_\varepsilon(x_0) \cap D|$. On the other hand, since $B_\varepsilon(x_i) \subset D_i$, it follows

$$|B_\varepsilon(x_0) \cap D| = \lim_{i\to\infty} |B_\varepsilon(x_i) \cap D| \geq \liminf_{i\to\infty} (|B_\varepsilon(x_i) \cap D_i| - |D_i \Delta D|) = \omega_n \varepsilon^n - \limsup_{i\to\infty} |D_i \Delta D| = \omega_n \varepsilon^n,$$

which is impossible.

If (iii) were false, then there would exist a subsequence of $x_i \in D_i \setminus D^\varepsilon$. Without loss of generality, assume $x_i \to x_0 \in \mathbb{R}^n \setminus D^\varepsilon$. For any i, by Remark 2.3, there is $c' > 0$ depending only on c and n such that $c'\varepsilon^n \leq |B_\varepsilon(x_i) \cap D_i|$. On the other hand, since $|B_\varepsilon(x_0) \cap D| = 0$, it follows

$$\liminf_{i\to\infty} |B_\varepsilon(x_i) \cap D_i| \leq \limsup_{i\to\infty} (|B_\varepsilon(x_i) \cap D| + |D \Delta D_i|) \leq |B_\varepsilon(x_0) \cap D| + \limsup_{i\to\infty} |D_i \Delta D| = 0,$$

which is a contradiction.

It remains to show $D \in \mathcal{D}_c$. Since $D_i \to D$ in L^1, for any $x \in \partial D$, there is $x_i \in D_i$ such that $x_i \to x$. Hence, for any $r > 0$, by Remark 2.3, we have

$$|B_r(x) \cap D| = \lim_i |B_r(x_i) \cap D| \geq \liminf_i |B_r(x_i) \cap D_i| - \limsup_i |D_i \Delta D| \geq c' r^n.$$

Hence $D \in \mathcal{D}_c$. □

The following remarks follow immediately from (i) and (iii) in the above lemma.

Remark 2.9. If D_i and D satisfy the same assumption as in Lemma 2.8 and if $\operatorname{int}(D) \neq \emptyset$, then $\operatorname{int}(D)$ is a domain. If in addition $|\operatorname{int}(D)| = |D|$, then $\operatorname{int}(D) \in \mathcal{D}_c$ and $D_i \to \operatorname{int}(D)$ in L^1.

For sets in $\mathcal{D}^c$, we have the following result, which is similar to Lemma 2.8.

Lemma 2.10. *If $D_i \in \mathcal{D}^c$ and $D_i \to D$ in L^1, and we identify D with its $\mathcal{L}^n$-equivalent set $\tilde{D}$ as in Proposition 2.7, then $D \in \mathcal{D}^c$. Moreover, for any $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that, for $i > N$, the following properties hold.*
(i) $D \subset D_i^\varepsilon$.
(ii) $(D_i)_\varepsilon \subset D$.
(iii′) $D_\varepsilon \subset D_i$.

3 Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. We start with the following two lemmas.

Lemma 3.1. *Let Ω be an M-uniform domain in $B_R \subset \mathbb{R}^n$ with $\operatorname{diam}(\Omega) \geq c_0 > 0$. Then there exist constants $\delta = \delta(M, n) \in (0, 1]$ and $C = C(c_0, M, R, n) > 0$ such that*

$$|(\partial\Omega)^r| \leq C r^\delta \quad \text{for all } r \in (0, 1]. \tag{3.1}$$

Lemma 3.2. *If Ω_i is a sequence of M-uniform domains in B_R such that $\operatorname{diam}(\Omega_i) \geq c > 0$ and $\Omega_i \to D$ in L^1, then there is an M-uniform domain Ω such that $\Omega_i \to \Omega$ in L^1.*

Lemma 3.1 is essentially proved in [23], where a more general result for porous domains is established. Here we present a simpler proof in the following for the reader's convenience. The ideas are from [23].

Proof of Lemma 3.1. Choose $k_0 \geq 1$ such that

$$2^{-k_0-1} \leq \frac{\min\{c_0, 1\}}{2} \leq 2^{-k_0}. \tag{3.2}$$

If $\frac{\min\{c_0,1\}}{2} \leq r \leq 1$, then

$$|(\partial\Omega)^r| \leq |B_{R+1}| \leq \frac{2|B_{R+1}|}{\min\{c_0, 1\}} r \leq \frac{2|B_{R+1}|}{\min\{c_0, 1\}} r^\delta \quad \text{for all } \delta \in (0, 1].$$

If $0 < r \leq \frac{\min\{c_0,1\}}{2}$, then we can find some $k \geq k_0$ such that $2^{-k-1} \leq r \leq 2^{-k}$.
It suffices to prove (3.1) for $r = 2^{-k}$, since it would then imply

$$|(\partial\Omega)^r| \leq C2^{-k\delta} = C(2^{-k-1})^{\frac{k\delta}{k+1}} \leq C r^{\frac{k\delta}{k+1}} \leq C r^{\frac{\delta}{2}}.$$

For any $x \in (\partial\Omega)^{2^{-k}}$, there exists $x_1 \in \partial\Omega$ such that $|x - x_1| < 2^{-k}$. Then, for any $k_0 \leq j \leq k$, by the choice of k_0 in (3.2), $\operatorname{diam}(\Omega) > 2^{-j+1}$ so that there exists $x_2 \in \partial B_{2^{-j+1}}(x_1) \cap \overline{\Omega}$. Let $\gamma \subset \overline{\Omega}$ be the path connecting x_1 and x_2 as in Definition 1.1. Let $y \in \partial B_{2^{-j}}(x_1) \cap \gamma$, and thus

$$d(y, \partial\Omega) \geq \frac{1}{M} \min\{|y - x_1|, |y - x_2|\} = \frac{2^{-j}}{M}. \tag{3.3}$$

We cover $B_R \setminus \partial\Omega$ by $\{B_{r_z}(z) : z \in B_R \setminus \partial\Omega,\ r_z = \frac{d(z,\partial\Omega)}{15}\} := \mathcal{B}_1$. By Vitalli's covering lemma, we can choose a countable pairwise disjoint subfamily $\mathcal{B}$ of $\mathcal{B}_1$ such that $B_R \setminus \partial\Omega \subset \bigcup_{B\in\mathcal{B}} 5B$. Hence $y \in B_{5r_z}(z)$ for some $B_{r_z}(z) \in \mathcal{B}$.

Clearly,

$$d(z, \partial\Omega) \le |z - x_1| \le |z - y| + |y - x_1| \le 5r_z + 2^{-j} = \frac{1}{3} d(z, \partial\Omega) + 2^{-j},$$

which implies

$$d(z, \partial\Omega) \le \frac{3}{2} 2^{-j}, \quad 5r_z \le 2^{-j-1}.$$

Therefore,

$$z \in B_{2^{-j+1}}(x_1) \setminus B_{2^{-j-1}}(x_1). \tag{3.4}$$

Notice that, by (3.3), it follows from $y \in B_{5r_z}(z)$ that $\frac{2^{-j}}{M} \le 20r_z$, and hence

$$|x - z| \le |x - x_1| + |x_1 - y| + |y - z| < 2^{-k} + 2^{-j} + 5r_z \le 2^{-j+1} + 5r_z \le (40M + 5)r_z \le 45Mr_z.$$

Therefore, $x \in B_{45Mr_z}(z)$.

So far, we have shown that, for any $x \in (\partial\Omega)^{2^{-k}}$ and $k_0 \le j \le k$, there is $z_j \in B_{2^{-j+1}}(x_1) \setminus B_{2^{-j-1}}(x_1)$ such that $x \in B_{45Mr_{z_j}}(z_j)$ and $B_{r_{z_j}}(z_j) \in \mathcal{B}$. Therefore, for all $x \in (\partial\Omega)^{2^{-k}}$, we have

$$\sum_{B\in\mathcal{B}} \chi_{45MB}(x) \ge \frac{k - k_0}{3}, \tag{3.5}$$

since by (3.4) each $B \in \mathcal{B}$ can be considered at most three times in order that $x \in 45MB$.

By the Hardy–Littlewood theorem, there is constant $c_n \ge 1$ such that, for any $p > 1$,

$$\|\mathcal{M}\phi\|_{L^p} \le c_n \Big(\frac{p}{p-1}\Big)^{\frac{1}{p}} \|\phi\|_{L^p},$$

where $\mathcal{M}\phi$ is the non-centered Hardy–Littlewood maximal function.

Let $\delta = \frac{1}{9(45M)^n c_n}$. By (3.5), we have

$$|(\partial\Omega)^{2^{-k}}| = 2^{-k\delta} \int\limits_{(\partial\Omega)^{2^{-k}}} 2^{k\delta} \le 2^{-k\delta} \int\limits_{(\partial\Omega)^{2^{-k}}} 2^{(k_0+3\sum_{B\in\mathcal{B}} \chi_{45MB}(x))\delta}\, dx \le 2^{-k\delta} 2^{k_0\delta} \int\limits_{B_R} \sum_{m=0}^{\infty} \frac{(3\delta \sum_{B\in\mathcal{B}} \chi_{45MB}(x))^m}{m!}\, dx.$$

For any nonnegative $\phi \in L^{\frac{m}{m-1}}$, $m > 1$, we have

$$\begin{aligned}\int \phi(x) \sum_{B\in\mathcal{B}} \chi_{45MB}(x)\, dx &\le (45M)^n \sum_{B\in\mathcal{B}} |B| \frac{1}{|45MB|} \int\limits_{45MB} \phi(x)\, dx \le (45M)^n \sum_{B\in\mathcal{B}} \int\limits_{B} \mathcal{M}\phi(x)\, dx \\ &\le (45M)^n \|M\phi\|_{L^{\frac{m}{m-1}}} \Big(\int \Big(\sum_{B\in\mathcal{B}} \chi_B(x)\Big)^m dx\Big)^{1/m} \le (45M)^n c_n m |B_{2R}|^{\frac{1}{m}} \|\phi\|_{L^{\frac{m}{m-1}}}.\end{aligned}$$

Hence, by duality, for $m > 1$, we obtain

$$\Big\|\sum_{B\in\mathcal{B}} \chi_{45MB}\Big\|_{L^m} \le (45M)^n c_n m |B_{2R}|^{\frac{1}{m}}. \tag{3.6}$$

It is straightforward to verify (3.6) for $m = 1$. Therefore,

$$\begin{aligned}|(\partial\Omega)^{2^{-k}}| &\le 2^{-k\delta} 2^{k_0\delta} |B_{2R}| \sum_{l=0}^{\infty} \frac{(3(45M)^n \delta c_n l)^l}{l!} \\ &\le 2^{-k\delta} 2^{k_0\delta} |B_{2R}| \sum_{l=0}^{\infty} \Big(\frac{e}{3}\Big)^l \quad \text{by Stirling's formula and the choice of } \delta \\ &= C(k_0, R, \delta, n) 2^{-k\delta} \le C(c_0, M, R, n) 2^{-k\delta} \quad \text{since } k_0 \text{ depends on } c_0.\end{aligned}$$

This completes the proof. □

Lemma 3.1 yields the following corollary.

Corollary 3.3. *Let Ω be an M-uniform domain in $B_R \subset \mathbb{R}^n$ with* $\operatorname{diam}(\Omega) \geq c_0 > 0$. *Then there exists a constant* $\delta = \delta(M, n) \in (0, 1]$ *such that, for any* $s \in (0, \delta)$,

$$[\chi_\Omega]_{W^{s,1}(B_R)} \leq C = C(M, n, R, s, c_0). \tag{3.7}$$

Proof. Let δ be as in Lemma 3.1. Then (3.7) follows from the estimate

$$\begin{aligned}\int_{B_R}\int_{B_R} \frac{|\chi_\Omega(x) - \chi_\Omega(y)|}{|x-y|^{n+s}}\,dy\,dx &= \int_{B_R}\int_0^{2R}\int_{\partial B_r(x)} \frac{|\chi_\Omega(x) - \chi_\Omega(y)|}{r^{n+s}}\,d\mathscr{H}^{n-1}(y)\,dr\,dx \\ &= \int_0^{2R}\int_{(\partial\Omega)^r}\int_{\partial B_r(x)} \frac{|\chi_\Omega(x) - \chi_\Omega(y)|}{r^{n+s}}\,d\mathscr{H}^{n-1}(y)\,dx\,dr \\ &\leq \int_0^{2R}\int_{(\partial\Omega)^r}\int_{\partial B_r(x)} \frac{1}{r^{n+s}}\,d\mathscr{H}^{n-1}(y)\,dx\,dr \\ &\leq \int_0^{2R} Cr^\delta r^{-s-1}\,dr \leq C(M, n, R, s, c_0) < \infty,\end{aligned}$$

where in the second equality we have used that if $x \notin (\partial\Omega)^r$ and $y \in B_r(x)$, then $\chi_\Omega(x) = \chi_\Omega(y)$. □

Next, we prove Lemma 3.2.

Proof of Lemma 3.2. Without loss of generality, we may assume $\operatorname{spt}\mu_D = \partial D$ as in Proposition 2.7. We first prove that $\operatorname{int}(D) \neq \emptyset$. Indeed, notice that, by Remark 2.5, each Ω_i contains a fixed ball of radius r_0 depending only on c_0, n and M. Therefore, for each Ω_i, if $\varepsilon < \frac{r_0}{2}$, then, by definition, $(\Omega_i)_\varepsilon$ contains a ball of radius $\frac{r_0}{2}$. By Lemma 2.8 (ii), D also contains a ball of radius $\frac{r_0}{2}$. In particular, $\operatorname{int}(D) \neq \emptyset$.

Now let $\Omega = \operatorname{int}(D)$. It suffices to show Ω is an M-uniform domain, since the L^1 convergence in the statement can then be directly deduced from Remark 2.2, Proposition 2.4 and the fact $\Omega \subset D \subset \overline{\Omega}$.

Fix any $x, y \in \Omega$. Then, for any given $N > 2M$, we may choose $0 < \varepsilon < \frac{1}{N}$ so small that

$$k\varepsilon < d(x, \partial\Omega) \leq (k+1)\varepsilon \quad \text{for some } k > \Big(1 + \frac{1}{M}\Big)(N+1),$$

and $|x - y| > 2(N+1)\varepsilon$. Since $\operatorname{int}(\Omega) \neq \emptyset$, it follows from Lemma 2.8 (i) and (iii) that $d_H(\Omega_i, \Omega) \to 0$. Hence we can find $x_i, y_i \in \Omega_i$, with $|x_i - x| < \varepsilon$, $|y_i - y| < \varepsilon$ for i large. By Lemma 2.8 (ii), we may choose i so large that

$$(\Omega_i)_\varepsilon \subset \Omega. \tag{3.8}$$

Also, we choose $\gamma_i \subset \Omega_i$ to be the rectifiable curve connecting x_i and y_i in Ω_i as in the definition of M-uniform domain. For any $p \in \gamma_i$, if $p \in B_{N\varepsilon}(x_i) \cup B_{N\varepsilon}(y_i)$, then clearly $p \in B_{(N+1)\varepsilon}(x) \cup B_{(N+1)\varepsilon}(y) \subset \Omega$. Moreover, this implies

$$d(p, \partial\Omega) \geq k\varepsilon - (N+1)\varepsilon > \frac{1}{M}(N+1)\varepsilon \geq \frac{1}{M}\min\{|p - x|, |p - y|\}. \tag{3.9}$$

Clearly, (3.9) also holds for any p on the line segment between x_i and x, and between y_i and y. If

$$p \notin B_{N\varepsilon}(x_i) \cup B_{N\varepsilon}(y_i),$$

then

$$d(p, \partial\Omega_i) \geq \frac{1}{M}\min\{|p - x_i|, |p - y_i|\} > \frac{N\varepsilon}{M},$$

and thus $p \in (\Omega_i)_{\frac{N\varepsilon}{M}} \subset (\Omega_i)_\varepsilon \subset \Omega \cap \Omega_i$. Moreover, let $r = d(p, \partial((\Omega_i)_\varepsilon))$. Then, by (3.8), we have $B_r(p) \subset \Omega$, so $d(p, \partial\Omega) \geq r = d(p, \partial((\Omega_i)_\varepsilon)) \geq d(p, \partial\Omega_i) - \varepsilon$. Therefore,

$$\frac{d(p, \partial\Omega)}{\min\{|p - x_i|, |p - y_i|\}} \geq \frac{d(p, \partial\Omega_i) - \varepsilon}{\min\{|p - x_i|, |p - y_i|\}} \geq \frac{1}{M} - \frac{\varepsilon}{N\varepsilon} \geq \frac{1}{M} - \frac{1}{N}.$$

Hence, by the choice of ε and N, it follows

$$d(p, \partial\Omega) \geq \left(\frac{1}{M} - \frac{1}{N}\right)(\min\{|p - x|, |p - y|\} - \varepsilon) \geq \left(\frac{1}{M} - \frac{1}{N}\right)(\min\{|p - x|, |p - y|\}) - \frac{1}{MN}. \tag{3.10}$$

Therefore, we may let γ^N be the curve that consists of the following three parts. The first part is a line segment starting from x to x_i, the second part is the curve γ_i found above, which starts from x_i to y_i, and the third part is a line segment starting from y_i to y.

It is clear from the discussion above that $\gamma^N \subset \Omega$ and γ^N starts from x to y. Moreover, from (3.9) and (3.10) and the choice of ε, we obtain that

$$\begin{aligned}\mathscr{H}^1(\gamma^N) &\leq M|x_i - y_i| + |x_i - x| + |y_i - y| \\ &\leq M|x - y| + (M+1)|x_i - x| + (M+1)|y_i - y| \leq M|x - y| + 2\frac{M+1}{N},\end{aligned}$$

and

$$d(p, \partial\Omega) \geq \left(\frac{1}{M} - \frac{1}{N}\right)\min\{|p - x|, |p - y|\} - \frac{1}{MN} \quad \text{for all } p \in \gamma^N.$$

Then, by the compactness of $(\overline{\Omega}, d_H)$ and since γ^N is connected, there is a compact connected set $E \subset \overline{\Omega}$ such that $d_H(\gamma^N, E) \to 0$ as $N \to \infty$. Then, by [15, Theorem 3.18],

$$\mathscr{H}^1(E) \leq \liminf_{N\to\infty} \mathscr{H}^1(\gamma^N) \leq M|x - y|.$$

Hence, by [15, Lemma 3.12], E is arc-wise connected so that we can choose a rectifiable curve $\gamma \subset E$ joining x and y. For any $p \in \gamma$, we can choose sequence $p_N \in \gamma^N$, $p_N \to p$. Since

$$d(p_N, \partial\Omega) \geq \left(\frac{1}{M} - \frac{1}{N}\right)\min\{|p_N - x|, |p_N - y|\} - \frac{1}{MN},$$

it follows by passing to the limit $N \to \infty$ that

$$d(p, \partial\Omega) \geq \frac{1}{M}\min\{|p - x|, |p - y|\},$$

which also implies $\gamma \subset \operatorname{int}(\Omega)$. Therefore, γ satisfies both properties in the definition of M-uniform domain, and Ω is M-uniform. By Corollary 2.9 and Proposition 2.4, Ω is a domain. This finishes the proof. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. By Corollary 3.3, the sequence χ_{Ω_i} is uniformly bounded in $W^{s,1}(B_R)$. By the compact embedding from $W^{s,1}(B_R)$ to space $L^q(B_R)$ with $1 \leq q \leq 1^* := \frac{n}{n-s}$, we conclude that there exists a subsequence of Ω_i that converges to a set $D \subset B_R$ in L^1. By Lemma 3.2, D is L^1 equivalent to an M-uniform domain. This finishes the proof. □

4 Uniform Poincaré inequality and existence of minimizer to (1.9)

In this section, we will apply Theorem 1.2 to deduce two uniform Poincaré inequalities via compactness argument, and then we will prove Theorem 1.4.

Theorem 4.1. *For any domain $\Omega \in \mathcal{M}_R$, there exists a constant $C > 0$ depending on M, R such that*

$$\int_\Omega u^2\,dx \leq C\int_\Omega |\nabla u|^2\,dx \quad \text{for all } u \in H^1(\Omega) \text{ with } \int_\Omega u\,dx = 0. \tag{4.1}$$

Proof. We divide the proof of (4.1) for $\Omega \in \mathcal{M}_R$ into two cases.

(i) If $\operatorname{diam}(\Omega) \geq 1$, then we argue by contradiction. Suppose there exist pairs (Ω_i, u_i) such that $\Omega_i \in \mathcal{M}_R$, $\operatorname{diam}(\Omega_i) \geq 1$, $u_i \in H^1(\Omega_i)$ satisfies $\int_{\Omega_i} u_i\,dx = 0$ and $\int_{\Omega_i} u_i^2\,dx = 1$, but $\int_{\Omega_i} |\nabla u_i|^2\,dx \to 0$ as $i \to \infty$. Let $\tilde{u}_i$ be an extension of u_i such that

$$\|\tilde{u}_i\|_{H^1(B_R)} \leq C(M, n)\|u_i\|_{H^1(\Omega_i)}.$$

Hence $\{\tilde{u}_i\}$ is a bounded sequence in $H^1(B_R)$. Hence we may assume that there exists $u \in H^1(B_R)$ such that $\tilde{u}_i \rightharpoonup u$ in $H^1(B_R)$ and $\tilde{u}_i \to u$ in $L^2(B_R)$. By Theorem 1.2, there is an M-uniform domain $\Omega \in \mathcal{M}_R$ such that $\Omega_i \to \Omega$ in L^1.

Since $\chi_{\Omega_i}\nabla\tilde{u}_i \rightharpoonup \chi_\Omega\nabla\tilde{u}$ weakly in L^2, by the lower semicontinuity property of weak convergence, we have

$$\int_\Omega |\nabla u|^2\,dx \le \liminf_{i\to\infty}\int_{\Omega_i}|\nabla u_i|^2\,dx = 0.$$

Hence $u \equiv c$ in Ω. On the other hand,

$$\begin{aligned}\Big|\int_{\Omega_i} u_i^2\,dx - \int_\Omega u^2\,dx\Big| &\le \Big|\int_{\Omega_i} u_i^2\,dx - \int_{\Omega_i} u^2\,dx\Big| + \Big|\int_{\Omega_i} u^2\,dx - \int_\Omega u^2\,dx\Big|\\ &\le \|\tilde{u}_i + u\|_{L^2(B_R)}\|\tilde{u}_i - u\|_{L^2(B_R)} + \int_{\Omega_i\Delta\Omega} u^2\,dx \to 0 \quad \text{as } i\to\infty.\end{aligned}$$

Hence

$$\int_\Omega u^2\,dx = 1. \tag{4.2}$$

Similarly, we have $\int_\Omega u\,dx = \lim_{i\to\infty}\int_{\Omega_i} u_i\,dx = 0$. Hence $c = 0$ and $\int_\Omega u^2\,dx = 0$. This contradicts (4.2). Therefore, we have proved (4.1).

(ii) If $\operatorname{diam}(\Omega) < 1$, then we may assume that $0 \in \Omega$. Hence we can choose a $0 < t < 1$ such that

$$\Omega_t := \frac{1}{t}\Omega \in \mathcal{M}_R \quad \text{with } \operatorname{diam}(\Omega_t) = 1.$$

For any $u \in H^1(\Omega)$ with $\int_\Omega u\,dx = 0$, from (i), we then have

$$\begin{aligned}\int_\Omega u^2(x)\,dx = t^n\int_{\Omega_t} u^2(tx)\,dx &\le Ct^n\int_{\Omega_t}|\nabla(u(tx))|^2\,dx = Ct^{n+2}\int_{\Omega_t}|\nabla u(tx)|^2\,dx = Ct^2\int_\Omega|\nabla u|^2\,dx\\ &\le C\int_\Omega|\nabla u(x)|^2\,dx\end{aligned}$$

since $0 < t < 1$. This finishes the proof. □

The second uniform Poincaré inequality has a slightly different form, which will be useful to prove the existence of minimization problem (1.9). See [18, 28, 31] for more background on traces and the Poincaré inequality on rough domains.

Theorem 4.2. *For any $\Omega \in \mathcal{M}_{R,c}$ with $P(\Omega) \le \Lambda$, there exists a constant $C > 0$ depending on M, c, Λ and R such that*

$$\int_\Omega u^2\,dx \le C\Big(\int_\Omega|\nabla u|^2\,dx + \Big(\int_{\partial^*\Omega}|u^*(x)|\,d\mathscr{H}^{n-1}\Big)^2\Big) \quad \textit{for all } u \in H^1(\Omega). \tag{4.3}$$

Proof. Suppose (4.3) were false. Then, by scaling, we may assume that there would exist pairs (Ω_i, u_i) such that $\Omega_i \in \mathcal{M}_{R,c}$, $P(\Omega_i) \le \Lambda$, $\operatorname{diam}(\Omega_i) \ge c$, $u_i \in H^1(\Omega_i)$ such that $\int_{\Omega_i} u_i^2 = 1$, but

$$\int_{\Omega_i}|\nabla u_i|^2\,dx + \Big(\int_{\partial^*\Omega}|u^*|\,d\mathscr{H}^{n-1}\Big)^2 \to 0 \quad \text{as } i\to\infty.$$

We may assume for convenience that $u_i \ge 0$. Let $\tilde{u}_i$ be an extension of u_i such that

$$\|\tilde{u}_i\|_{H^1(B_R)} \le C(M,n)\|u_i\|_{H^1(\Omega_i)}.$$

Hence $\{\tilde{u}_i\}$ is a bounded sequence in $H^1(B_R)$. Let $u \in H^1(B_R)$ be the weak limit of $\tilde{u}_i$ in $H^1(B_R)$ and $\tilde{u}_i \to u$ in $L^2(B_R)$. By Theorem 1.2 and lower semicontinuity of sets of finite perimeter, there is an M-uniform domain $\Omega \in \mathcal{M}_{R,c}$ with $P(\Omega) \le \Lambda$ such that $\Omega_i \to \Omega$ in L^1.

As in the proof of Theorem 4.1, we have that

$$\int_{\Omega} |\nabla u|^2 \, dx \le \liminf_{i\to\infty} \int_{\Omega_i} |\nabla u_i|^2 \, dx = 0,$$

and thus $u \equiv c$ in Ω for some constant c. Also,

$$\int_{\Omega} u^2 \, dx = 1. \tag{4.4}$$

Now let $\bar{u}_i = \tilde{u}_i \chi_{\Omega_i}$ and $\bar{u} = u\chi_{\Omega}$. By [2, Theorem 3.84] and the structure of the BV function, we know that $\bar{u}_i, u \in \mathrm{SBV}(\mathbb{R}^n)$, with

$$J_{\bar{u}_i} = \partial^* \Omega_i \cap \{u_i^* > 0\} \quad \text{and} \quad J_u = \partial^* \Omega \cap \{u^* > 0\}.$$

Here J_u denotes the measure theoretical jump part of a BV function u.

We let w^- and w^+ denote the measure theoretical interior and exterior trace of a BV function w on $\partial^*\Omega$ respectively. Since $\mathcal{H}^{n-1}(\partial^*\Omega_i) \le \Lambda$ and $\nabla \tilde{u}_i \chi_{\Omega_i} \rightharpoonup \nabla \tilde{u} \chi_{\Omega}$ weakly in $L^2(B_R)$, we can apply [6, Theorem 2.3 and Theorem 2.12] to obtain

$$\int_{\partial^*\Omega} u^* \, d\mathcal{H}^{n-1} = \int_{J_u} |u^- - u^+| \, d\mathcal{H}^{n-1} \le \liminf_{i\to\infty} \int_{J_{\bar{u}_i}} |\bar{u}_i^- - \bar{u}_i^+| \, d\mathcal{H}^{n-1} = \liminf_{i\to\infty} \int_{\partial^*\Omega_i} u_i^*.$$

Hence $\int_{\partial^*\Omega} u^* \, d\mathcal{H}^{n-1} = 0$ and $u \equiv 0$ in Ω. This contradicts (4.4). □

Now we are ready to give a proof of Theorem 1.4.

Proof of Theorem 1.4. Let (u_i, Ω_i) be a minimizing sequence, and we may assume that u_i is a minimizer of $\mathcal{J}_m(\cdot, \Omega_i)$ among all $H^1(\Omega_i)$ functions. From $\mathcal{J}_m(u_i, \Omega_i) \le \mathcal{J}_m(0, \Omega_i) = 0$, we deduce that

$$\begin{aligned}\int_{\Omega_i} |\nabla u_i|^2 \, dx + \frac{1}{2m}\Big(\int_{\partial\Omega_i} u_i \, d\mathcal{H}^{n-1}\Big)^2 &\le \int_{\Omega_i} f u_i \, dx \le \varepsilon \int_{\Omega_i} u_i^2 \, dx + C_\varepsilon \int_{\Omega} f^2 \, dx \\ &\le C\varepsilon\Big(\int_{\Omega} |\nabla u_i|^2 \, dx + \Big(\int_{\partial^*\Omega} |u_i^*| \, d\mathcal{H}^{n-1}\Big)^2\Big) + C_\varepsilon \int_{\Omega} f^2 \, dx,\end{aligned}$$

where we have used Theorem 4.2. By choosing a small $\varepsilon > 0$, this implies that

$$\sup_i \Big(\int_{\Omega_i} |\nabla u_i|^2 \, dx + \int_{\partial\Omega_i} u_i \, d\mathcal{H}^{n-1}\Big) < \infty. \tag{4.5}$$

Hence the infimum of $\mathcal{J}_m > -\infty$. Moreover, by Theorem 4.2 and (4.5), $\sup_i \|u_i\|_{H^1(\Omega_i)} < \infty$. Now we can repeat the same argument as in the proof of Theorem 4.2 to conclude that there exists a $(u, \Omega) \in \mathcal{A}$ such that

$$\mathcal{J}_m(u, \Omega) \le \liminf_{i\to\infty} \mathcal{J}_m(u_i, \Omega_i).$$

The proof is completed. □

5 Existence of minimizers in SBV

In this section, we will extend the existence results of the previous section to the setting of SBV and prove Theorem 1.5. The argument of our proof is similar to that in [12].

Proof of Theorem 1.5. We prove it by the direct method of calculus of variation.

Claim 1. *$\mathcal{J}$ is bounded from below on $\mathbb{S}$.*

For any $u \in \mathcal{S}$, since $\operatorname{supp} u \subset D$ and $\mathcal{H}^{n-1}(J_u \cap \partial D) = 0$, we have the following Sobolev type inequality [29, Theorem 4.10]:

$$\|u\|_{L^{\frac{n}{n-1}}(D)} \le C|Du|(D). \tag{5.1}$$

From (5.1), Young's inequality and the fact that $t^2 > t - 1$, we can derive

$$\begin{aligned}
\mathcal{J}(u) &\ge \frac{1}{4}\int_D |\nabla u|^2\,dx + \frac{1}{4m}\Big(\int_{J_u} (|u^+| + |u^-|)\,d\mathcal{H}^{n-1}\Big)^2 \\
&\qquad + \frac{1}{4}\int_D (|\nabla u| - 1)\,dx + \frac{1}{4m}\Big(\int_{J_u} (|u^+| + |u^-|)\,d\mathcal{H}^{n-1} - 1\Big) - \int_D fu\,dx \\
&\ge \frac{1}{4}\int_D |\nabla u|^2\,dx + \frac{1}{4m}\Big(\int_{J_u} (|u^+| + |u^-|)\,d\mathcal{H}^{n-1}\Big)^2 \\
&\qquad + C\Big(\int_D |\nabla u|\,dx + \int_{J_u} (|u^+ - u^-|)\,d\mathcal{H}^{n-1}\Big) - C - \int_D fu\,dx \\
&= \frac{1}{4}\int_D |\nabla u|^2\,dx + \frac{1}{4m}\Big(\int_{J_u} (|u^+| + |u^-|)\,d\mathcal{H}^{n-1}\Big)^2 + C|Du|(D) - C - \int_D fu\,dx \\
&\ge \frac{1}{4}\int_D |\nabla u|^2\,dx + \frac{1}{4m}\Big(\int_{J_u} (|u^+| + |u^-|)\,d\mathcal{H}^{n-1}\Big)^2 + C|Du|(D) - C - \varepsilon\|u\|_{L^{\frac{n}{n-1}}(D)} - C(\varepsilon)\|f\|_{L^n(D)} \\
&\ge -C - C\|f\|_{L^n(D)},
\end{aligned} \tag{5.2}$$

provided ε is chosen sufficiently small. Hence the functional $\mathcal{J}$ is bounded from below, and we can find a minimizing sequence $\{u_i\}$ in $\mathcal{S}$ such that

$$\lim_{i\to\infty} \mathcal{J}(u_i) = \inf_{u\in\mathcal{S}} \mathcal{J}(u) > -\infty.$$

Claim 2. *There exists $u \in \mathrm{SBV}(D)$ such that, after taking a subsequence, $u_i \rightharpoonup u$ in BV.*

From the penultimate inequality of (5.2), we have

$$\begin{aligned}
\sup_i \|u_i\|_{\mathrm{BV}(D)} = \sup_i (|Du_i|(D) + \|u_i\|_{L^1(D)}) &\le C\sup_i (\mathcal{J}(u_i) + C + C\|f\|_{L^n(D)}) < \infty, \\
\sup_i \Big(\int_D |\nabla u_i|^2\,dx + \int_{J_{u_i}} (|u_i^+| + |u_i^-|)\,d\mathcal{H}^{n-1}\Big) &\le C\sup_i (\mathcal{J}(u_i) + C + C\|f\|_{L^n(D)}) < \infty.
\end{aligned} \tag{5.3}$$

By the compactness theorem of BV functions [2, Theorem 3.23], there exists a subsequence $\{u_{i_k}\}$ and $u \in \mathrm{BV}(D)$ such that $u_{i_k} \rightharpoonup u$ in $\mathrm{BV}(D)$, i.e.

$$\begin{cases} u_{i_k} \to u & \text{in } L^1(D), \\ Du_{i_k} \overset{*}{\rightharpoonup} Du & \text{in } \mathcal{M}(D). \end{cases} \tag{5.4}$$

For every $\varepsilon > 0$, let $u_{i_k}^\varepsilon := \max\{u_{i_k}, \varepsilon\}$, $u^\varepsilon := \max\{u, \varepsilon\}$. Then we have

$$u_{i_k}^\varepsilon \rightharpoonup u^\varepsilon \text{ in } \mathrm{BV}(D). \tag{5.5}$$

From (5.3), we have

$$\sup_k \int_D |\nabla u_{i_k}^\varepsilon|^2 = \sup_k \int_D |\nabla u_{i_k} \chi_{\{u_{i_k} > \varepsilon\}}|^2 \le \sup_k \int_D |\nabla u_{i_k}|^2 < \infty. \tag{5.6}$$

Moreover, from the Chebyshev inequality, we have

$$\sup_k \mathcal{H}^{n-1}(J_{u_{i_k}^\varepsilon}) \le \sup_k \frac{1}{\varepsilon} \int_{J_{u_{i_k}}} (|u_{i_k}^+| + |u_{i_k}^-|)\,d\mathcal{H}^{n-1} \le \frac{C}{\varepsilon}, \tag{5.7}$$

where we use that fact that $J_{u_{i_k}^\varepsilon} \subset J_{u_{i_k}} \cap \{u_{i_k} > \varepsilon\}$.

Now, from (5.5), (5.6) and (5.7), we can apply the SBV compactness theorem [2, Theorem 4.7] to conclude that $u^\varepsilon \in \mathrm{SBV}(D)$, and

$$\begin{cases} \nabla u^\varepsilon_{i_k} \rightharpoonup \nabla u^\varepsilon & \text{in } L^1(D), \\ D^j u^\varepsilon_{i_k} \overset{\star}{\rightharpoonup} D^j u^\varepsilon & \text{in } \mathcal{M}(D), \end{cases}$$

where D^j denotes the jump part of the distributional gradient Du. Moreover,

$$\int_D |\nabla u^\varepsilon|^2 \le \liminf_{k\to\infty} \int_D |\nabla u^\varepsilon_{i_k}|^2 \le \liminf_{k\to\infty} \int_D |\nabla u_{i_k}|^2. \tag{5.8}$$

Since $\nabla u^\varepsilon = \nabla u \chi_{\{u>\varepsilon\}} \to \nabla u$ a.e. in D as $\varepsilon \to 0$, by Fatou's lemma, we have that

$$\int_D |\nabla u|^2 \le \liminf_{\varepsilon\to 0} \int_D |\nabla u^\varepsilon|^2 \le \sup_k \int_D |\nabla u_{i_k}|^2 < \infty,$$

and this implies $\nabla u \in L^2(D)$. From the dominated convergence theorem we have that

$$\nabla u^\varepsilon \to \nabla u \quad \text{in } L^2(D), \quad \text{as } \varepsilon \to 0. \tag{5.9}$$

For the jump part of u, since $u \in \mathrm{BV}(\mathbb{R}^n)$, we get

$$\int_{J_u} |u^+ - u^-| \, d\mathscr{H}^{n-1} < \infty. \tag{5.10}$$

Notice that

$$D^j u^\varepsilon = ((u^\varepsilon)^+ - (u^\varepsilon)^-)\nu_u \mathscr{H}^{n-1}\lfloor_{J_u}. \tag{5.11}$$

By (5.10), (5.11) and the dominated convergence theorem, we have

$$D^j u^\varepsilon \to D^j u \quad \text{in } \mathcal{M}(D), \quad \text{as } \varepsilon \to 0. \tag{5.12}$$

Since both convergence of (5.9) and (5.12) are strong, the Cantor part $D^c u$ of Du vanishes. In fact, for any open set A,

$$|Du|(A) \le \liminf_{\varepsilon\to 0} |Du^\varepsilon|(A) = \liminf_{\varepsilon\to 0} \Big(\int_A |\nabla u^\varepsilon| \, dx + |D^j u^\varepsilon|(A) \Big) = \int_A |\nabla u| \, dx + |D^j u|(A),$$

which implies $|D^c u|(A) = 0$. Hence $D^c u \equiv 0$ and $u \in \mathrm{SBV}(\mathbb{R}^n)$. From (5.4), we can derive that $|\operatorname{supp} u \setminus D| = 0$, and $|\{u > 0\}| = V_0$.

Claim 3. *The lower semicontinuity property holds for functional $\mathcal{J}$.*

From (5.8) and (5.9), we can conclude that

$$\int_D |\nabla u|^2 \le \lim_{k\to\infty} \int_D |\nabla u_{i_k}|^2. \tag{5.13}$$

For any open set $A \subset \mathbb{R}^n$, in view of the bound estimate (5.6), we can apply the lower semicontinuity result from [6, Theorem 2.12] to $\{u^\varepsilon_{i_k}\}$ to obtain

$$\int_{J_{u^\varepsilon} \cap A} (|(u^\varepsilon)^+| + |(u^\varepsilon)^-|) \, d\mathscr{H}^{n-1} \le \liminf_{k\to\infty} \int_{J_{u^\varepsilon_{i_k}} \cap A} (|(u^\varepsilon_{i_k})^+| + |(u^\varepsilon_{i_k})^-|) \, d\mathscr{H}^{n-1}. \tag{5.14}$$

Passing the ε to 0 and applying the monotone convergence theorem to the left-hand side of (5.14) gives

$$\int_{J_u \cap A} (|u^+| + |u^-|) \, d\mathscr{H}^{n-1} \le \liminf_{k\to\infty} \int_{J_{u_{i_k}} \cap A} (|u^+_{i_k}| + |u^-_{i_k}|) \, d\mathscr{H}^{n-1}.$$

Choose $A = \mathbb{R}^n \setminus D$. We then get $\mathscr{H}^{n-1}(J_u \setminus D) = 0$ and hence $u \in \mathcal{S}$. From (5.4), (5.13) and (5.14), we can conclude that

$$\mathcal{J}(u) \le \liminf_k \mathcal{J}(u_{i_k}) = \inf_{u\in\mathcal{S}} \mathcal{J}(u),$$

which entails u is a minimizer of the problem. □

6 Some properties on smooth critical points

In this section, we will show that smooth solutions are stationary critical points.

For a bounded C^2-domain $\Omega \subset \mathbb{R}^n$, since $\mathcal{J}_m(\cdot, \Omega)\colon H^1(\Omega) \mapsto \mathbb{R}$ is convex, it is readily seen in [10] that there exists a unique critical point, denoted as u_Ω, of

$$\mathcal{J}_m(v, \Omega) := \frac{1}{2}\int_\Omega |\nabla v|^2\, dx + \frac{1}{2m}\Big(\int_{\partial\Omega} |v|\, d\sigma\Big)^2 - \int_\Omega v\, dx \tag{6.1}$$

over $v \in H^1(\Omega)$. In fact, u_Ω is a minimal point of $\mathcal{J}_m(\cdot, \Omega)$ over $v \in H^1(\Omega)$. Since $\mathcal{J}_m(|u_\Omega|, \Omega) \le \mathcal{J}_m(u_\Omega, \Omega)$, we conclude that $u_\Omega \ge 0$. Moreover, we have the following proposition on the regularity of $\overline{\Omega}$.

Proposition 6.1. *If $\Omega \subset \mathbb{R}^n$ is a C^2 bounded domain and $u \in H^1(\Omega)$ is a minimizer of $\mathcal{J}_m(\cdot, \Omega)$ over $H^1(\Omega)$, then $u \in W^{1,p}(\Omega)$ for any $1 \le p < \infty$ and*

$$\max\{\|u\|_{W^{1,p}(\Omega)}, \|(\nabla u)^*\|_{L^p(\partial\Omega)}\} \le C(m, p, \|\Omega\|_{C^2}). \tag{6.2}$$

Proof. For any $\varepsilon > 0$, consider $\mathcal{J}_m^\varepsilon(\cdot, \Omega)$ an ε-regularization of $\mathcal{J}_m(\cdot, \Omega)$, which is defined by

$$\mathcal{J}_m^\varepsilon(v, \Omega) = \frac{1}{2}\int_\Omega |\nabla v|^2\, dx + \frac{1}{2m}\Big(\int_{\partial\Omega} \sqrt{v^2 + \varepsilon^2}\, d\sigma\Big)^2 - \int_\Omega v\, dx.$$

Let $v_\varepsilon \in H^1(\Omega)$ be a minimizer of $\mathcal{J}_m^\varepsilon(\cdot, \Omega)$, whose existence is standard. Then $v_\varepsilon \ge 0$ in Ω, and direct calculations imply that v_ε is a weak solution to the following Neumann boundary value problem:

$$\begin{cases} -\Delta v_\varepsilon = 1 & \text{in } \Omega, \\ \dfrac{\partial v_\varepsilon}{\partial \nu} = g_\varepsilon := \Big(\dfrac{1}{m}\displaystyle\int_{\partial\Omega} \sqrt{v_\varepsilon^2 + \varepsilon^2}\, d\sigma\Big)\dfrac{v_\varepsilon}{\sqrt{v_\varepsilon^2 + \varepsilon^2}} & \text{on } \partial\Omega. \end{cases}$$

It is easy to see that $\mathcal{J}_m^\varepsilon(v_\varepsilon, \Omega) \le \mathcal{J}_m^\varepsilon(1, \Omega) \le C(m, |\partial\Omega|, |\Omega|)$ for all $0 < \varepsilon \le 1$. This, combined with the Poincaré inequality, implies that

$$\int_\Omega |\nabla v_\varepsilon|^2 + \Big(\int_{\partial\Omega} |v_\varepsilon|\Big)^2 \le C(m, |\partial\Omega|, |\Omega|) \quad \text{for all } 0 < \varepsilon \le 1,$$

and hence $\|v_\varepsilon\|_{H^1(\Omega)} \le C(m, |\partial\Omega|, |\Omega|)$ for all $0 < \varepsilon \le 1$. Since $|g_\varepsilon| \le \frac{1}{m}\int_{\partial\Omega} \sqrt{1 + v_\varepsilon^2}$ on $\partial\Omega$, this implies that $g_\varepsilon \in L^\infty(\partial\Omega)$, and $\|g_\varepsilon\|_{L^\infty(\partial\Omega)} \le C(m, |\partial\Omega|, |\Omega|)$ for all $0 < \varepsilon \le 1$. Therefore, we can apply the standard elliptic theory to conclude that $v_\varepsilon \in W^{1,p}(\Omega)$ for any $1 \le p < \infty$, and $\|v_\varepsilon\|_{W^{1,p}(\Omega)} \le C(m, p, \|\Omega\|_{C^2})$ for all $0 < \varepsilon \le 1$. In fact, we have the stronger estimate, namely the L^p-norm of the non-tangential maximal function of ∇v_ε can be bounded by that of g_ε, i.e. $\|(\nabla v_\varepsilon)^*\|_{L^p(\partial\Omega)} \le C(m, p, \|\Omega\|_{C^2})\|g_\varepsilon\|_{L^p(\partial\Omega)}$ for all $1 < p < \infty$. Hence we may assume, after taking a possible subsequence, that there exists $v \in W^{1,p}(\Omega)$, $p \in (1, \infty)$, such that $v_\varepsilon \rightharpoonup v$ in $W^{1,p}(\Omega)$ for all $1 \le p < \infty$. Now we want to show that v is also a minimizer of $\mathcal{J}_m(\cdot, \Omega)$. In fact, for any function $w \in H^1(\Omega)$, we have that $\mathcal{J}_m^\varepsilon(v_\varepsilon, \Omega) \le \mathcal{J}_m^\varepsilon(w, \Omega)$. Since $v_\varepsilon \rightharpoonup v$ in $H^1(\Omega)$, it follows from the lower semicontinuity that

$$\mathcal{J}_m(v, \Omega) \le \liminf_{\varepsilon\to 0} \mathcal{J}_m^\varepsilon(v_\varepsilon, \Omega) \le \liminf_{\varepsilon\to 0} \mathcal{J}_m^\varepsilon(w, \Omega) = \mathcal{J}_m(w, \Omega).$$

Since $\mathcal{J}_m(\cdot, \Omega)$ is convex over $H^1(\Omega)$, there is a unique minimizer of $\mathcal{J}_m(\cdot, \Omega)$ in $H^1(\Omega)$. Hence $u \equiv v$ in Ω. This proves (6.2). □

It follows from Proposition 6.1 and the Sobolev embedding theorem that $u \in C^\alpha(\overline{\Omega})$ for any $0 < \alpha < 1$. Hence, by direct calculations, we obtain that $u = u_\Omega \ge 0$ is a weak solution to the following boundary value problem:

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = -\dfrac{1}{m}\displaystyle\int_{\partial\Omega} u\, d\sigma & \text{on } \partial\Omega \cap \{x : u(x) > 0\}, \\ \dfrac{\partial u}{\partial \nu} \ge -\dfrac{1}{m}\displaystyle\int_{\partial\Omega} u\, d\sigma & \text{on } \partial\Omega \cap \{x : u(x) = 0\}. \end{cases} \tag{6.3}$$

It is readily seen that $u \not\equiv 0$ on $\partial\Omega$. The following lemma indicates that any nonnegative weak solution of (6.3) also minimizes $\mathfrak{J}_m(\,\cdot\,,\Omega)$.

Lemma 6.2. *For any bounded C^2-domain $\Omega \subset \mathbb{R}^n$, if $u \in H^1(\Omega) \cap C^1(\overline{\Omega})$ is a nonnegative weak solution of* (6.3), *then $\mathfrak{J}_m(u,\Omega) \le \mathfrak{J}_m(v,\Omega)$ for all $v \in H^1(\Omega)$.*

Proof. For any $v \in H^1(\Omega)$, multiplying (6.3) by $u - v$ and integrating over Ω, we obtain

$$\int_\Omega |\nabla u|^2\,dx - \int_\Omega u\,dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} u\,d\sigma = \int_\Omega \nabla u \cdot \nabla v\,dx - \int_\Omega v\,dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v\,d\sigma. \tag{6.4}$$

From the second equation of (6.3), we see that

$$-\int_{\partial\Omega} \frac{\partial u}{\partial \nu} u\,d\sigma = \Big(\frac{1}{m}\int_{\partial\Omega} u\,d\sigma\Big)\int_{\partial\Omega} u = \frac{1}{m}\Big(\int_{\partial\Omega} u\,d\sigma\Big)^2.$$

On the other hand, we have

$$\begin{aligned}
-\int_{\partial\Omega} \frac{\partial u}{\partial \nu} v\,d\sigma &= -\int_{\partial\Omega\cap\{u(x)>0\}} \frac{\partial u}{\partial \nu} v\,d\sigma - \int_{\partial\Omega\cap\{u(x)=0\}} \frac{\partial u}{\partial \nu} v\,d\sigma \\
&= \Big(\frac{1}{m}\int_{\partial\Omega} u\,d\sigma\Big)\int_{\partial\Omega\cap\{u(x)>0\}} v\,d\sigma - \int_{\partial\Omega\cap\{u(x)=0\}} \frac{\partial u}{\partial \nu} v\,d\sigma \\
&= \Big(\frac{1}{m}\int_{\partial\Omega} u\,d\sigma\Big)\int_{\partial\Omega} v\,d\sigma - \int_{\partial\Omega\cap\{u(x)=0\}} \Big(\frac{\partial u}{\partial \nu} + \frac{1}{m}\int_{\partial\Omega} u\,d\sigma\Big) v\,d\sigma \\
&= \Big(\frac{1}{m}\int_{\partial\Omega} u\,d\sigma\Big)\int_{\partial\Omega\cap\{v(x)>0\}} v\,d\sigma + \Big(\frac{1}{m}\int_{\partial\Omega} u\,d\sigma\Big)\int_{\partial\Omega\cap\{v(x)\le 0\}} v\,d\sigma \\
&\quad - \Big(\frac{1}{m}\int_{\partial\Omega} u\,d\sigma\Big)\int_{\substack{\partial\Omega\cap\{u(x)=0\}\\ \cap\{v(x)\le 0\}}} v\,d\sigma \\
&\quad - \int_{\substack{\partial\Omega\cap\{u(x)=0\}\\ \cap\{v(x)\le 0\}}} \frac{\partial u}{\partial \nu} v\,d\sigma - \int_{\substack{\partial\Omega\cap\{u(x)=0\}\\ \cap\{v(x)>0\}}} \Big(\frac{\partial u}{\partial \nu} + \frac{1}{m}\int_{\partial\Omega} u\,d\sigma\Big) v\,d\sigma \\
&\le \Big(\frac{1}{m}\int_{\partial\Omega} u\,d\sigma\Big)\int_{\partial\Omega} |v|\,d\sigma - \int_{\substack{\partial\Omega\cap\{u(x)=0\}\\ \cap\{v(x)\le 0\}}} \frac{\partial u}{\partial \nu} v\,d\sigma - \int_{\substack{\partial\Omega\cap\{u(x)=0\}\\ \cap\{v(x)>0\}}} \Big(\frac{\partial u}{\partial \nu} + \frac{1}{m}\int_{\partial\Omega} u\,d\sigma\Big) v\,d\sigma.
\end{aligned}$$

It follows from the third equation of (6.3) that

$$\Big(\frac{\partial u}{\partial \nu}(x) + \frac{1}{m}\int_{\partial\Omega} u\,d\sigma\Big) v(x) \ge 0 \quad \text{for all } x \in \partial\Omega \cap \{u(x) = 0\} \cap \{v(x) > 0\},$$

and hence

$$\int_{\substack{\partial\Omega\cap\{u(x)=0\}\\ \cap\{v(x)>0\}}} \Big(\frac{\partial u}{\partial \nu} + \frac{1}{m}\int_{\partial\Omega} u\,d\sigma\Big) v\,d\sigma \ge 0.$$

Since $u \in C^1(\overline{\Omega})$ satisfies $u > 0$ in Ω, it follows that $\frac{\partial u}{\partial \nu}(x) \le 0$ on $\partial\Omega \cap \{u(x) = 0\}$, and hence

$$\int_{\substack{\partial\Omega\cap\{u(x)=0\}\\ \cap\{v(x)\le 0\}}} \frac{\partial u}{\partial \nu} v\,d\sigma \ge 0.$$

Thus we obtain

$$-\int_{\partial\Omega} \frac{\partial u}{\partial \nu} v\,d\sigma \le \Big(\frac{1}{m}\int_{\partial\Omega} u\,d\sigma\Big)\int_{\partial\Omega} |v|\,d\sigma,$$

and hence

$$\begin{aligned}\int_\Omega \nabla u\cdot\nabla v\,dx - \int_\Omega v\,dx - \int_{\partial\Omega}\frac{\partial u}{\partial \nu}v\,d\sigma &\le \int_\Omega \nabla u\cdot\nabla v\,dx - \int_\Omega v\,dx + \Big(\frac{1}{m}\int_{\partial\Omega} u\,d\sigma\Big)\int_{\partial\Omega}|v|\,d\sigma\\ &\le \frac12\int_\Omega|\nabla u|^2\,dx + \frac12\int_\Omega|\nabla v|^2\,dx - \int_\Omega v\,dx\\ &\quad + \frac{1}{2m}\Big(\int_{\partial\Omega} u\,d\sigma\Big)^2 + \frac{1}{2m}\Big(\int_{\partial\Omega}|v|\,d\sigma\Big)^2.\end{aligned}$$

Substituting this into (6.4) yields that $\mathfrak{J}_m(u,\Omega)\le\mathfrak{J}_m(v,\Omega)$. □

For $m>0$, it follows from the discussion above that if $u\in H^1(\Omega)$ is a critical point of $\mathfrak{J}_m(\cdot,\Omega)$, then $u\ge 0$ in $\overline{\Omega}$. If, in addition, $u>0$ in $\overline{\Omega}$, then it follows from (6.3) that u solves

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega,\\ \dfrac{\partial u}{\partial \nu} = -\dfrac{1}{m}\displaystyle\int_{\partial\Omega} u\,d\sigma & \text{on } \partial\Omega.\end{cases}$$

Thus it follows from the standard elliptic theory that $u\in C^{1,\beta}(\overline{\Omega})$ for all $0<\beta<1$. However, the following example shows that there exists a bounded C^2-domain Ω such that any minimizer $u\in H^1(\Omega)$ to $\mathfrak{J}_m(\cdot,\Omega)$ has zero points on $\partial\Omega$.

Example 6.3. For $n=2$ and $\Omega=\{x\in\mathbb{R}^2 : 1<|x|<2\}$, if $0<m<3\pi-4\pi\ln 2$, then

$$u(x) = -\frac14|x|^2 + c_1\ln|x| + c_2 \quad \text{for } x\in\Omega,$$

with

$$c_1 = \frac{m+3\pi}{2m+4\pi\ln 2},\qquad c_2 = \frac{2m-(m-\pi)\ln 2}{2m+4\pi\ln 2},$$

is the unique minimizer of $\mathfrak{J}_m(\cdot,\Omega)$ over $H^1(\Omega)$.

Proof. Notice that $\partial\Omega=\partial B_1\cup\partial B_2$. It is easy to see that $u>0$ in $\Omega\cup\partial B_1$ and $u=0$ on ∂B_2, and it satisfies

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega,\\ \dfrac{\partial u}{\partial \nu} = -\dfrac{1}{m}\displaystyle\int_{\partial B_1} u & \text{on } \partial B_1,\\ \dfrac{\partial u}{\partial \nu} > -\dfrac{1}{m}\displaystyle\int_{\partial\Omega} u & \text{on } \partial B_2.\end{cases}$$

From Lemma 6.2, u is a minimizer of $\mathfrak{J}_m(\cdot,\Omega)$ in $H^1(\Omega)$. □

Proposition 6.4. *If $u\in W^{2,2}(\Omega)$ is a critical point of $\mathfrak{J}_m(\cdot,\Omega)$, then it is also critical with respect to the domain variation, i.e.*

$$\frac{d}{dt}\Big|_{t=0}\mathfrak{J}_m(u^t,\Omega)=0,$$

where $u^t(x)=u(F(t,x))$, and $F(\cdot,\cdot): (-\delta,\delta)\times\overline{\Omega}\mapsto\overline{\Omega}$ is a C^1-family of C^2-diffeomorphism satisfying

$$\begin{cases} F(0,x)=x & \textit{for all } x\in\overline{\Omega},\\ F(t,x)\in\partial\Omega & \textit{for all } (x,t)\in\partial\Omega\times(-\delta,\delta).\end{cases}$$

Proof. Define the deformation vector field $\eta(x)=\frac{d}{dt}\big|_{t=0}F(t,x)$ for $x\in\overline{\Omega}$. Then

$$\eta(x)\in T_x(\partial\Omega)\quad\text{or}\quad \eta(x)\cdot\nu(x)=0\quad\text{for all } x\in\partial\Omega.$$

By direct calculations, we have

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\Big(\frac{1}{2}\int_\Omega |\nabla u^t|^2\,dx\Big) &= -\frac{1}{2}\int_\Omega |\nabla u|^2\operatorname{div}\eta\,dx + \int_\Omega u_i u_j \eta^i_j\,dx \\
&= -\frac{1}{2}\int_\Omega |\nabla u|^2\operatorname{div}\eta\,dx + \int_{\partial\Omega}\eta\cdot\nabla u\frac{\partial u}{\partial\nu}\,d\sigma - \int_\Omega \Delta u(\eta\cdot\nabla u)\,dx - \frac{1}{2}\int_\Omega \eta\cdot\nabla(|\nabla u|^2)\,dx \\
&= -\frac{1}{2}\int_\Omega \operatorname{div}(|\nabla u|^2\eta)\,dx + \int_{\partial\Omega}\eta\cdot\nabla u\frac{\partial u}{\partial\nu}\,d\sigma - \int_\Omega \Delta u(\eta\cdot\nabla u)\,dx \\
&= -\frac{1}{2}\int_{\partial\Omega}|\nabla u|^2\eta\cdot\nu\,d\sigma + \int_\Omega \eta\cdot\nabla u\,dx + \int_{\partial\Omega}\eta\cdot\nabla u\frac{\partial u}{\partial\nu}\,d\sigma \\
&= \int_\Omega \eta\cdot\nabla u\,dx + \int_{\partial\Omega}\eta\cdot\nabla_{\tan}u\frac{\partial u}{\partial\nu}\,d\sigma,
\end{aligned}$$

where we have used the first equation of (6.3), and $\nabla_{\tan}u = (\mathbb{I}_n - \nu\otimes\nu)\nabla u$.

$$\frac{d}{dt}\Big|_{t=0}\Big\{\frac{1}{2m}\Big(\int_{\partial\Omega}u^t\,d\sigma\Big)^2\Big\} = \frac{1}{m}\int_{\partial\Omega}u\,d\sigma\int_{\partial\Omega}\eta\cdot\nabla_{\tan}u\,d\sigma.$$

It is readily seen that

$$\frac{d}{dt}\Big|_{t=0}\Big(-\int_\Omega u^t\,dx\Big) = -\int_\Omega \eta\cdot\nabla u\,dx.$$

Putting these identities together, we obtain that

$$\frac{d}{dt}\Big|_{t=0}\mathcal{J}_m(u^t,\Omega) = \int_{\partial\Omega}\eta\cdot\nabla_{\tan}u\Big(\frac{\partial u}{\partial\nu} + \frac{1}{m}\int_{\partial\Omega}u\,d\sigma\Big)\,d\sigma = \int_{\partial\Omega\cap\{u>0\}}\eta\cdot\nabla_{\tan}u\Big(\frac{\partial u}{\partial\nu} + \frac{1}{m}\int_{\partial\Omega}u\,d\sigma\Big)\,d\sigma = 0.$$

This completes the proof. □

Definition 6.5. Given a bounded C^2-domain $\Omega\subset\mathbb{R}^n$, let $u = u_\Omega \in H^1(\Omega)$ be the unique minimizer of (6.1). We say that (u,Ω) is a critical point of $\mathcal{J}_m(\cdot,\cdot)$ if either $I(t) = \mathcal{J}_m(u_{\Omega(t)},\Omega(t))$ is not differentiable at $t=0$, or

$$\frac{d}{dt}\Big|_{t=0}\mathcal{J}_m(u_{\Omega(t)},\Omega(t)) = 0, \tag{6.5}$$

where $\Omega(t) = \{F(t,x) : x\in\Omega\}$ and $u_{\Omega(t)}$ is the unique minimizer of $\mathcal{J}_m(\cdot,\Omega(t))$ over $H^1(\Omega(t))$. Here

$$F(t,x)\colon(-\delta,\delta)\times\overline{\Omega}\mapsto\mathbb{R}^n$$

is any C^1-family of C^2-volume preserving diffeomorphism, that is generated by a vector field $\eta\in C^2(\overline{\Omega},\mathbb{R}^n)$, i.e.

$$\frac{dF}{dt}(t,x) = \eta(F(t,x)),\quad F(0,x) = x,\quad \text{for all } x\in\Omega,\ -\delta<t<\delta.$$

Here $(u_{\Omega(0)},\Omega(0)) = (u,\Omega)$.

Theorem 6.6. *For $m>0$ and a bounded C^2-domain $\Omega\subset\mathbb{R}^n$, let u_Ω be the unique minimizer of $\mathcal{J}_m(\cdot,\Omega)$ over $H^1(\Omega)$. If u_Ω is positive in $\overline{\Omega}$, then (u_Ω,Ω) is a critical point of $\mathcal{J}_m(\cdot,\cdot)$ if and only if the following identity holds:*

$$\frac{1}{2}|\nabla_{\tan}u_\Omega|^2 - u_\Omega - \frac{1}{2}\Big(\frac{1}{m}\int_{\partial\Omega}u_\Omega\Big)^2 + \Big(\frac{1}{m}\int_{\partial\Omega}u_\Omega\Big)u_\Omega H \equiv \text{constant}\quad \text{on } \partial\Omega, \tag{6.6}$$

where H denotes the mean curvature of $\partial\Omega$. In particular, for any ball $B_R\subset\mathbb{R}^n$ with radius R, (u_{B_R},B_R) is a critical point of $\mathcal{J}_m(\cdot,\cdot)$.

Proof. For simplicity, denote $u = u_\Omega$. Since $u\in C(\overline{\Omega})$ is positive, it follows that u solves (6.5) so that we have $u\in C^{1,\alpha}(\overline{\Omega})\cap W^{2,2}(\Omega)$. Hence there exists $\delta_0>0$ such that $u\geq\delta_0$ in $\overline{\Omega}$. For a small $0<\delta_1\ll\delta_0$ and an open set $U\supset\overline{\Omega}$, let $F(t,x)\colon(-\delta_1,\delta_1)\times U\mapsto\mathbb{R}^n$ be a C^1-family of C^2-volume preserving diffeomorphism, generated by a vector field $\eta \in C^2(U, \mathbb{R}^n)$. It is readily seen that $\Omega(t) = F(t)(\Omega)$, $-\delta_1 < t < \delta_1$, is a C^1-family of bounded C^2-domains. By an argument similar to that of Proposition 6.1, we can show that $u(t) \equiv u_{\Omega(t)}(F(t,\cdot)) \to u$ in $C^0(\overline{\Omega})$ as $t \to 0$ so that there exists $0 < \delta_2 < \delta_1$ such that $u(t)(y) \geq \frac{\delta_0}{2}$ for $y \in \overline{\Omega(t)}$ and $t \in (-\delta_2, \delta_2)$. Hence $u(t)$, $-\delta_2 < t < \delta_2$, solves

$$\begin{cases} -\Delta u(t) = 1 & \text{in } \Omega(t), \\ \dfrac{\partial}{\partial \nu} u(t) = -\dfrac{1}{m} \displaystyle\int_{\partial\Omega(t)} u(t)(y)\, d\sigma & \text{on } \partial\Omega(t). \end{cases} \tag{6.7}$$

Applying Proposition 6.1 again, we have that, for any $1 < p < \infty$,

$$\|u(t)\|_{W^{2,2}(\Omega(t))} + \|u(t)\|_{W^{1,p}(\Omega(t))} \leq C(p), \quad t \in (-\delta_2, \delta_2).$$

This implies $\mathcal{J}_m(u(t), \Omega(t)) \in C^1((-\delta_2, \delta_2))$.

It follows from $|\Omega(t)| = |\Omega|$ for $-\delta_2 < t < \delta_2$ that $\int_\Omega \operatorname{div} \eta = 0$. Now we calculate $\frac{d}{dt}\mathcal{J}_m(u(t), \Omega(t))$ for $t \in (-\delta_2, \delta_2)$. We claim that

$$\frac{d}{dt}\mathcal{J}_m(u(t), \Omega(t)) = \int_{\partial\Omega(t)} \Big[\frac{1}{2}|\nabla_{\tan} u(t)|^2 - \frac{1}{2}|\nabla_\nu u(t)|^2 - u(t) + \Big(\frac{1}{m}\int_{\partial\Omega(t)} u(t)\Big) u(t) H(t)\Big] \eta \cdot \nu \, d\sigma$$

for all $t \in (-\delta_2, \delta_2)$. Here $H(t)$ denotes the mean curvature of $\partial\Omega(t)$, and $\nabla_{\tan} f = (\mathbb{I}_n - \nu \otimes \nu)\nabla f$ denotes the tangential derivative of f on $\partial\Omega(t)$.

To simplify the proof, denote $u(t,x) = u(t)(x)$, and set $v(t,x) \equiv \frac{\partial}{\partial t} u(t,x)$, $x \in \Omega(t)$. Notice that $\Omega = \Omega(0)$ and $u_\Omega(x) = u(0,x)$, $x \in \Omega$. Recall the formula [20, Corollary 5.2.8]

$$\frac{d}{dt}\int_{\Omega(t)} f(t,y)\, dy = \int_{\Omega(t)} \frac{\partial f}{\partial t}(t,y)\, dy + \int_{\partial\Omega(t)} f(t,y)\eta(y)\cdot\nu(t,y)\, d\sigma \tag{6.8}$$

for any $f \in C^1(\{(t,x) : t \in (-\delta_2, \delta_2),\ x \in \Omega(t)\})$, where $\nu(t,\cdot)$ denotes the outward unit normal of $\partial\Omega(t)$. Applying (6.8), we can calculate

$$\begin{aligned} I_1(t) &\equiv \frac{d}{dt}\int_{\Omega(t)} \frac{1}{2}|\nabla u(t,x)|^2\, dx \\ &= \int_{\Omega(t)} \nabla u(t,x)\cdot\nabla v(t,x)\, dx + \int_{\partial\Omega(t)} \frac{1}{2}|\nabla u(t,x)|^2 \eta(x)\cdot\nu(t,x)\, d\sigma \\ &= -\int_{\Omega(t)} \Delta u(t,x) v(t,x)\, dx + \int_{\partial\Omega(t)} v(t,x)\partial_\nu u(t,x)\, d\sigma + \int_{\partial\Omega(t)} \frac{1}{2}|\nabla u(t,x)|^2\eta(x)\cdot\nu(t,x)\, d\sigma, \\ I_3(t) &\equiv \frac{d}{dt}\int_{\Omega(t)} u(t,x)\, dx = \int_{\Omega(t)} v(t,x)\, dx + \int_{\partial\Omega(t)} u(t,x)\eta(x)\cdot\nu(t,x)\, d\sigma. \end{aligned}$$

Also, recall the formula [20, Proposition 5.4.18]

$$\frac{d}{dt}\int_{\partial\Omega(t)} f(t,x)\, dx = \int_{\partial\Omega(t)} \Big(\frac{\partial f}{\partial t}(t,x) + \frac{\partial f}{\partial \nu}(t,x)\eta(x)\cdot\nu(t,x)\Big)\, d\sigma + \int_{\partial\Omega(t)} f(t,x)H(t)(x)\eta(x)\cdot\nu(t,x)\, d\sigma \tag{6.9}$$

for any $f \in C^1(\{(t,x) : t \in (-\delta_2, \delta_2),\ x \in \Omega(t)\})$. Applying (6.9) and (6.7), we find

$$\begin{aligned} I_2(t) &\equiv \frac{d}{dt}\Big\{\frac{1}{2m}\Big(\int_{\partial\Omega_t} u(t,x)\, d\sigma\Big)^2\Big\} \\ &= \Big(\frac{1}{m}\int_{\partial\Omega(t)} u(t,x)\, d\sigma\Big)\int_{\partial\Omega(t)} \Big(v(t,x) + \Big(\frac{\partial u}{\partial \nu}(t,x) + u(t,x)H(t,x)\Big)\eta(x)\cdot\nu(t,x)\Big)\, d\sigma \\ &= -\int_{\partial\Omega(t)} \frac{\partial u}{\partial \nu}(t,x)\Big[v(t,x) + \Big(\frac{\partial u}{\partial \nu}(t,x) + u(t,x)H(t,x)\Big)\eta(x)\cdot\nu(t,x)\Big]\, d\sigma, \end{aligned}$$

where $H(t,x) = H(t)(x)$ denotes the mean curvature of $\partial\Omega(t)$ at $x \in \partial\Omega(t)$.

Adding $I_1(t)$, $I_2(t)$ and $-I_3(t)$ together, and applying the first equation of (6.7), we obtain that

$$\begin{aligned}\frac{d}{dt}\mathfrak{J}_m(u(t),\Omega(t)) &= I_1(t)+I_2(t)-I_3(t)\\ &= \int_{\Omega(t)}(-\Delta u(t,x)-1)v(t,x)\,dx\\ &\quad+\int_{\partial\Omega(t)}\Big(\frac{1}{2}|\nabla u(t,x)|^2-\Big|\frac{\partial u}{\partial \nu}\Big|^2(t,x)+u(t,x)\\ &\qquad\qquad-\frac{\partial u}{\partial \nu}(t,x)u(t,x)H(t,x)\Big)\eta(x)\cdot\nu(t,x)\,d\sigma\\ &= \int_{\partial\Omega(t)}\Big(\frac{1}{2}|\nabla_{\tan}u(t,x)|^2-\frac{1}{2}\Big|\frac{\partial u}{\partial \nu}\Big|^2(t,x)+u(t,x)\\ &\qquad\qquad-\frac{\partial u}{\partial \nu}(t,x)u(t,x)H(t,x)\Big)\eta(x)\cdot\nu(t,x)\,d\sigma.\end{aligned}\tag{6.10}$$

Thus, by setting $t=0$ and applying the second equation of (6.7), we obtain that

$$\frac{d}{dt}\Big|_{t=0}\mathfrak{J}_m(u(t),\Omega(t)) = \int_{\partial\Omega}\Big(\frac{1}{2}|\nabla_{\tan}u|^2-\frac{1}{2}\Big|\frac{\partial u}{\partial \nu}\Big|^2+u-\Big(\frac{1}{m}\int_{\partial\Omega}u\,d\sigma\Big)uH\Big)\eta(x)\cdot\nu\,d\sigma.\tag{6.11}$$

Note that, for any given C^1-family of volume preserving C^2-diffeomorphism maps $F(t,x)\colon(-\delta_1,\delta_1)\times\overline{\Omega}\mapsto\mathbb{R}^n$ for some $\delta_1>0$, it is necessary that the velocity field η satisfies $\int_{\partial\Omega}\eta\cdot\nu\,d\sigma=0$. Substituting such an η into (6.11), we see that (6.6) holds if and only if (u_Ω,Ω) is a critical point of $\mathfrak{J}_m(\cdot,\cdot)$.

Recall that, when $\Omega=B_R$, the unique critical point of $\mathfrak{J}_m(\cdot,B_R)$ is given by

$$u_{B_R}(x)=\frac{R^2-|x|^2}{2n}+\frac{m}{n^2\omega_nR^{n-2}},\quad x\in B_R,\tag{6.12}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Since u_{B_R} is smooth and positive in $\overline{B_R}$, and satisfies (6.6), it follows that (u_{B_R},B_R) is a critical point of $\mathfrak{J}_m(\cdot,\cdot)$. □

7 Stability of (u_{B_R},B_R)

It follows from Theorem 6.6 that, for any $R>0$, (u_{B_R},B_R) is a critical point for $\mathfrak{J}_m(\cdot,\cdot)$ for any $m>0$. In this section, we will prove Theorem 1.7, namely, (u_{B_R},B_R) is a stable critical point of $\mathfrak{J}_m(\cdot,\cdot)$.

Proof of Theorem 1.7. It follows from the discussion in the previous section that there exists $\delta_0>0$ such that $u(t,x)=u_{\Omega(t)}(x)$ is positive, satisfies (6.7) and is smooth in $\overline{\Omega}(t)$ for $t\in(-\delta,\delta)$. Hence, by formula (6.10), we have that, for $t\in(-\delta,\delta)$,

$$\begin{aligned}\frac{d}{dt}\mathfrak{J}_m(u(t),\Omega(t)) &= \int_{\partial\Omega(t)}\Big[\frac{1}{2}|\nabla u|^2(t,x)-\Big|\frac{\partial u}{\partial\nu}\Big|^2(t,x)-u(t,x)-\frac{\partial u}{\partial\nu}(t,x)u(t,x)H(t,x)\Big]\eta(x)\cdot\nu(t,x)\,d\sigma\\ &= \mathrm{I}(t)+\mathrm{II}(t)+\mathrm{III}(t)+\mathrm{IV}(t).\end{aligned}\tag{7.1}$$

To simplify the presentation, set

$$v(x)=\frac{\partial u}{\partial t}(0,x),\quad u_0(x)=u(0,x),\quad x\in B_R,$$

and $\zeta(x)=\eta(x)\cdot\nu(x)$ for $x\in\partial B_R$. From the volume constraint $|\Omega(t)|=|B_R|$ for $t\in(-\delta,\delta)$, we claim that

$$\int_{\partial B_R}\zeta(x)\,d\sigma=\int_{B_R}\operatorname{div}\eta(x)\,dx=0,\tag{7.2}$$

$$\int_{\partial B_R}\zeta(x)\operatorname{div}\eta(x)\,d\sigma=\int_{B_R}\operatorname{div}(\operatorname{div}\eta\,\eta)\,dx=0.\tag{7.3}$$

To see this, notice that, since $|\Omega(t)| = \int_{B_R} JF(t,x)\,dx$ is constant, we have

$$\frac{d}{dt}\Big|_{t=0} \int_{B_R} JF(t,x)\,dx = \frac{d^2}{dt^2}\Big|_{t=0} \int_{B_R} JF(t,x)\,dx = 0,$$

while by direct calculations, we have

$$\frac{d}{dt} JF(t,x) = (\operatorname{div} \eta \circ F(t,x)) JF(t,x)$$

and

$$\frac{d^2}{dt^2} JF(t,x) = (\operatorname{div} \eta \circ F(t,x))^2 + (\nabla \operatorname{div} \eta \circ F(t,x))(\eta \circ F(t,x)) JF(t,x).$$

Thus we obtain

$$\begin{cases} \int_{B_R} \operatorname{div} \eta(x)\,dx = 0, \\ \int_{B_R} \operatorname{div}(\operatorname{div} \eta \eta)(x)\,dx = \int_{B_R} ((\operatorname{div} \eta)^2 + \eta \nabla \operatorname{div} \eta)(x)\,dx = 0 \end{cases}$$

so that (7.2) and (7.3) hold.

From (6.12), we see that

$$u_0 = \frac{m}{n^2 \omega_n R^{n-2}} \quad \text{and} \quad \nabla u_0(x) = -\frac{x}{n} \quad \text{on } \partial B_R; \qquad \frac{\partial u_0}{\partial \nu} = -\frac{R}{n} \quad \text{on } \partial B_R.$$

Applying (6.9), we have

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \Big(\frac{1}{m} \int_{\partial\Omega(t)} u(t,x)\,d\sigma\Big) &= \frac{1}{m} \int_{\partial B_R} \Big(v(x) + \frac{\partial u_0}{\partial \nu}(x)\zeta(x) + u_0(x)H(x)\zeta(x)\Big)\,d\sigma \\ &= \frac{1}{m} \int_{\partial B_R} v(x)\,d\sigma + \Big(\frac{n-1}{n^2 \omega_n R^{n-1}} - \frac{R}{nm}\Big) \int_{\partial B_R} \zeta(x)\,d\sigma = \frac{1}{m} \int_{\partial B_R} v(x)\,d\sigma, \end{aligned} \tag{7.4}$$

where we have used $H = \frac{n-1}{R}$ on ∂B_R.

Now we want to show that v solves the following boundary value problem in B_R:

$$\begin{cases} -\Delta v = 0 & \text{in } B_R, \\ \dfrac{\partial v}{\partial \nu} = \dfrac{\zeta}{n} & \text{on } \partial B_R. \end{cases} \tag{7.5}$$

To see (7.5), let $\phi \in C_0^\infty(B_{R+1})$. Then, by (6.8), we have

$$0 = \frac{d}{dt}\Big|_{t=0} \int_{\Omega(t)} (\Delta u(t,x) + 1)\phi(x)\,dx = \int_{B_R} \Delta v(x)\phi(x)\,dx + \int_{\partial B_R} (\Delta u_0 + 1)\phi(x)\zeta(x)\,d\sigma = \int_{B_R} \Delta v(x)\phi(x)\,dx,$$

where we have used the fact that $\Delta u_0 + 1 = 0$ on ∂B_R. Since ϕ is arbitrary, we conclude that $\Delta v = 0$ in B_R. To show v satisfies the boundary condition of (7.5) (second equation), we apply (7.4) and (6.9) and proceed as follows:

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \int_{\partial\Omega(t)} \phi(x)\Big[v(t,x) \cdot \nabla u(t,x) + \Big(\frac{1}{m} \int_{\partial\Omega(t)} u(t,y)\,d\sigma\Big)\Big]\,d\sigma \\ &= \int_{\partial B_R} \phi(x)\Big(\frac{x}{R} \cdot \nabla v(x) + \frac{\partial v}{\partial t}(0,x) \cdot \nabla u_0 + \Big[\frac{x}{R} \cdot \nabla\Big(\frac{x}{|x|}\Big) \cdot \nabla u_0 + \frac{x}{R} \otimes \frac{x}{R} : \nabla^2 u_0\Big]\zeta(x)\Big)\,d\sigma \\ &\quad + \Big(\frac{1}{m} \int_{\partial B_R} v(x)\,d\sigma\Big) \int_{\partial B_R} \phi(x)\,d\sigma + \int_{\partial B_R} \phi(x)\Big(\frac{\partial u_0}{\partial \nu} + \Big(\frac{1}{m} \int_{\partial B_R} u_0(x)\,d\sigma\Big)\Big)H(x)\zeta(x)\,d\sigma \\ &= \int_{\partial B_R} \phi(x)\Big(\frac{\partial v(x)}{\partial \nu} - \frac{1}{n}\zeta(x) + \frac{1}{m} \int_{\partial B_R} v(x)\,d\sigma\Big), \end{aligned} \tag{7.6}$$

where we have used the following facts:

$$\begin{aligned}
\Big\langle \frac{\partial v}{\partial t}(0,x), \nabla u_0(x) \Big\rangle &= -\frac{R}{n}\Big\langle \frac{\partial v}{\partial t}(0,x), v(0,x) \Big\rangle = 0 &&\text{on } \partial B_R,\\
\frac{x}{R}\cdot\nabla\Big(\frac{x}{|x|}\Big)\cdot\nabla u_0 &= -\frac{1}{n}\frac{x}{R}\cdot\nabla\Big(\frac{x}{|x|}\Big)\cdot x = 0 &&\text{on } \partial B_R,\\
\frac{x}{R}\otimes\frac{x}{R} : \nabla^2 u_0 &= -\frac{1}{n}\frac{x}{R}\otimes\frac{x}{R} : I_n = -\frac{1}{n} &&\text{on } \partial B_R
\end{aligned}$$

and

$$\frac{\partial u_0}{\partial \nu} + \frac{1}{m}\int_{\partial B_R} u_0(x)\,d\sigma = 0 \quad \text{on } \partial B_R.$$

It follows from (7.6) that

$$\frac{\partial v}{\partial \nu} = \frac{\zeta}{n} - \frac{1}{m}\int_{\partial B_R} v(x)\,d\sigma \quad \text{on } \partial B_R. \tag{7.7}$$

Since $\Delta v = 0$ in B_R, we have $\int_{\partial B_R} \frac{\partial v}{\partial \nu}\,d\sigma = 0$, which, combined with (7.7) and $\int_{\partial B_R} \zeta = 0$, implies that

$$\frac{1}{m}\int_{\partial B_R} v(x)\,d\sigma = 0. \tag{7.8}$$

Thus v solves (7.5). From (7.8) and (7.4), we also have that

$$\frac{d}{dt}\Big|_{t=0}\Big(\frac{1}{m}\int_{\partial\Omega(t)} u(t,x)\,d\sigma\Big) = 0. \tag{7.9}$$

Next we want to compute the second-order variation based on (7.1). First, applying (6.9), we have

$$\begin{aligned}
\mathrm{I}'(0) &= \frac{d}{dt}\Big|_{t=0}\int_{\partial\Omega(t)} \frac{1}{2}|\nabla u|^2(t,x)\eta(x)\cdot\nu(t,x)\,d\sigma\\
&= \int_{\partial B_R}\Big(\nabla u_0(x)\cdot\nabla v(x)\zeta(x) + \frac{1}{2}|\nabla u_0(x)|^2\eta(x)\cdot\frac{\partial \nu}{\partial t}(0,x)\Big)\,d\sigma\\
&\quad + \int_{\partial B_R}\Big(\eta(x)\cdot\nabla^2 u_0(x)\cdot\nabla u_0(x)\eta(x)\cdot\nu(x) + \frac{1}{2}|\nabla u_0(x)|^2\eta\cdot\nabla(\eta(x)\cdot\nu(x))\Big)\,d\sigma\\
&\quad + \int_{\partial B_R}\frac{1}{2}|\nabla u_0(x)|^2 H(x)(\eta(x)\cdot\nu(x))^2\,d\sigma,
\end{aligned} \tag{7.10}$$

where we have used the fact that $\nu(x) = \nu(0,x)$ for $x \in \partial B_R$.

Since $\langle \frac{\partial \nu}{\partial t}(0,x), \nu(x)\rangle = 0$ and $\eta(x) = \zeta(x)\nu(x)$ on ∂B_R, we see that

$$\int_{\partial B_R}\frac{1}{2}|\nabla u_0(x)|^2\eta(x)\cdot\frac{\partial \nu}{\partial t}(0,x)\,d\sigma = 0. \tag{7.11}$$

Since $\nu(x) = \frac{x}{R}$ and $\nabla u_0(x) = -\frac{x}{n}$ on ∂B_R, by (7.5), we see that

$$\int_{\partial B_R}\nabla u_0(x)\cdot\nabla v(x)\zeta(x)\,d\sigma = -\frac{R}{n^2}\int_{\partial B_R}\zeta^2(x)\,d\sigma. \tag{7.12}$$

Direct calculations yield

$$\begin{aligned}
\int_{\partial B_R}\eta(x)\cdot\nabla^2 u_0(x)\cdot\nabla u_0(x)\eta(x)\cdot\nu(x)\,d\sigma &= \int_{\partial B_R}\eta^i(x)\Big(-\frac{|x|^2}{2n}\Big)_{ij}\Big(-\frac{|x|^2}{2n}\Big)_j\zeta(x)\,d\sigma\\
&= \frac{1}{n^2}\int_{\partial B_R}(\eta(x)\cdot x)\zeta(x)\,d\sigma = \frac{R}{n^2}\int_{\partial B_R}\zeta^2(x)\,d\sigma.
\end{aligned} \tag{7.13}$$

Notice that, on ∂B_R, we have the formula

$$\eta \cdot \nabla(\eta \cdot \nu) = \zeta\langle \nu, \nabla\zeta\rangle = \zeta \operatorname{div}(\zeta\nu) - \zeta^2 \operatorname{div}\nu = \zeta \operatorname{div}\eta - \zeta^2 H. \tag{7.14}$$

Thus we obtain that

$$\int_{\partial B_R} \frac{1}{2}|\nabla u_0(x)|^2 \eta \cdot \nabla(\eta(x)\cdot\nu(x))\,d\sigma = \int_{\partial B_R} \frac{1}{2}|\nabla u_0(x)|^2 (\zeta(x)\operatorname{div}\eta(x) - \zeta^2(x)H(x))\,d\sigma. \tag{7.15}$$

Substituting (7.11), (7.12), (7.13), (7.15) into (7.10) and applying (7.3), we obtain that

$$\mathrm{I}'(0) = \int_{\partial B_R} \frac{1}{2}|\nabla u_0(x)|^2 \zeta(x)\operatorname{div}\eta(x)\,d\sigma = \frac{R^2}{2n^2}\int_{\partial B_R} \zeta(x)\operatorname{div}\eta(x)\,d\sigma = 0. \tag{7.16}$$

Next, by (7.3) and (6.9), we compute

$$\begin{aligned}
\mathrm{III}'(0) &= -\frac{d}{dt}\Big|_{t=0} \int_{\partial\Omega(t)} u(t,x)\eta(x)\cdot\nu(t,x)\,d\sigma \\
&= -\int_{\partial B_R} (v(x)\zeta(x) + u_0(x)\eta(x)\cdot\frac{\partial\nu}{\partial t}(0,x))\,d\sigma \\
&\quad - \int_{\partial B_R} \Big(\frac{\partial u_0(x)}{\partial\nu}\zeta^2(x) + u_0(x)\eta\cdot\nabla(\eta(x)\cdot\nu(x))\Big)\,d\sigma - \int_{\partial B_R} u_0(x)\zeta^2(x)H(x)\,d\sigma \\
&= -\int_{\partial B_R} \Big(v(x)\zeta(x) - \frac{R}{n}\zeta^2(x)\Big)\,d\sigma - \int_{\partial B_R} u_0(x)\zeta(x)\operatorname{div}\eta(x)\,d\sigma \\
&= \frac{R}{n}\int_{\partial B_R} \zeta^2(x)\,d\sigma - \int_{\partial B_R} v(x)\zeta(x)\,d\sigma,
\end{aligned} \tag{7.17}$$

where we have used the fact $\eta(x)\cdot\frac{\partial\nu}{\partial t}(0,x) = 0$, (7.14), and (7.16) on ∂B_R.

Recall that the mean curvature of $\partial\Omega(t)$ satisfies (see Huisken [22])

$$\frac{\partial H}{\partial t}(t,x) = -\Delta_{\partial\Omega(t)}\eta(x)\cdot\nu(t,x) - |A(t,x)|^2\eta(x)\cdot\nu(t,x), \quad x \in \partial\Omega(t), \tag{7.18}$$

where $\Delta_{\partial\Omega(t)}$ is the Laplace operator on $\partial\Omega(t)$ and A is the second fundamental form of $\partial\Omega(t)$.

Applying (7.9), (6.9), (7.14) and (7.15), we can compute

$$\begin{aligned}
\mathrm{II}'(0) &= -\frac{d}{dt}\Big|_{t=0}\Big[\Big(\frac{1}{m}\int_{\partial\Omega(t)} u(t,x)\,d\sigma\Big)^2 \int_{\partial\Omega(t)} \eta(x)\cdot\nu(t,x)\,d\sigma\Big] \\
&= -\Big(\frac{1}{m}\int_{\partial B_R} u_0\Big)^2 \frac{d}{dt}\Big|_{t=0}\int_{\partial\Omega(t)} \eta(x)\cdot\nu(t,x)\,d\sigma \\
&= -\frac{R^2}{n^2}\int_{\partial B_R}\Big(\eta(x)\cdot\frac{\partial\nu}{\partial t}(0,x) + \eta(x)\cdot\nabla(\eta(x)\cdot\nu(x)) + H(x)\zeta^2(x)\Big)\,d\sigma \\
&= -\frac{R^2}{n^2}\int_{\partial B_R}\Big(\eta(x)\cdot\frac{\partial\nu}{\partial t}(0,x) + (\zeta(x)\operatorname{div}\eta(x) - \zeta^2(x)H(x)) + H(x)\zeta^2(x)\Big)\,d\sigma \\
&= -\frac{R^2}{n^2}\int_{\partial B_R} \zeta(x)\operatorname{div}\eta(x)\,d\sigma = 0.
\end{aligned} \tag{7.19}$$

Applying (7.9), (6.9), (7.18) and (7.14), and using

$$\frac{1}{m}\int_{\partial B_R} u_0(x)\,d\sigma = \frac{R}{n}, \quad |A(x)|^2 = \frac{n-1}{R^2} \quad \text{for } x \in \partial B_R,$$

we can compute

$$\begin{aligned}
\mathrm{IV}'(0) &= \frac{d}{dt}\Big|_{t=0} \int\limits_{\partial\Omega(t)} \Big(\frac{1}{m}\int\limits_{\partial\Omega(t)} u(t,y)\,d\sigma\Big) u(t,x)H(t,x)\eta(x)\cdot\nu(t,x)\,d\sigma \\
&= \int\limits_{\partial B_R} \frac{d}{dt}\Big|_{t=0}\Big(\frac{1}{m}\int\limits_{\partial\Omega(t)} u(t,y)\,d\sigma\Big) u_0(x)H(x)\zeta(x)\,d\sigma \\
&\quad + \Big(\frac{1}{m}\int\limits_{\partial B_R} u_0(x)\,d\sigma\Big)\int\limits_{\partial B_R} (v(x)+\eta(x)\cdot\nabla u_0(x))H(x)\zeta(x)\,d\sigma \\
&\quad + \Big(\frac{1}{m}\int\limits_{\partial B_R} u_0(x)\,d\sigma\Big)\int\limits_{\partial B_R} u_0(x)(-\Delta_{\partial B_R}\zeta(x) - |A(x)|^2\zeta(x))\zeta(x)\,d\sigma \\
&\quad + \Big(\frac{1}{m}\int\limits_{\partial B_R} u_0(x)\,d\sigma\Big)\int\limits_{\partial B_R} u_0(x)H(x)(\zeta(x)\operatorname{div}\eta(x) - \zeta^2(x)H(x))\,d\sigma \\
&\quad + \Big(\frac{1}{m}\int\limits_{\partial B_R} u_0(x)\,d\sigma\Big)\int\limits_{\partial B_R} u_0(x)H^2(x)\zeta^2(x)\,d\sigma \\
&= \frac{R}{n}\Big[\frac{n-1}{R}\int\limits_{\partial B_R} v(x)\zeta(x)\,d\sigma - \frac{n-1}{n}\int\limits_{\partial B_R}\zeta^2(x)\,d\sigma \\
&\qquad + \frac{m}{n^2\omega_n R^{n-2}}\int\limits_{\partial B_R}(-\Delta_{\partial B_R}\zeta(x) - |A(x)|^2\zeta(x))\zeta(x)\,d\sigma\Big] \\
&= -\frac{(n-1)R}{n^2}\int\limits_{\partial B_R}\zeta^2(x)\,d\sigma + \frac{n-1}{n}\int\limits_{\partial B_R} v(x)\zeta(x)\,d\sigma \\
&\quad + \frac{m}{n^3\omega_n R^{n-3}}\int\limits_{\partial B_R}\Big(|\nabla_{\tan}\zeta(x)|^2 - \frac{n-1}{R^2}\zeta^2(x)\Big)\,d\sigma.
\end{aligned} \tag{7.20}$$

Therefore, by adding (7.10), (7.19), (7.17) and (7.20) together, we obtain

$$\begin{aligned}
\frac{d^2}{dt^2}\Big|_{t=0}\mathcal{J}_m(u(t),\Omega(t)) &= \mathrm{I}'(0)+\mathrm{II}'(0)+\mathrm{III}'(0)+\mathrm{IV}'(0) = \mathrm{I}'(0)+\mathrm{IV}'(0) \\
&= \frac{R}{n^2}\int\limits_{\partial B_R}\zeta^2(x)\,d\sigma - \frac{1}{n}\int\limits_{\partial B_R} v(x)\zeta(x)\,d\sigma \\
&\quad + \frac{m}{n^3\omega_n R^{n-3}}\int\limits_{\partial B_R}\Big(|\nabla_{\tan}\zeta(x)|^2 - \frac{n-1}{R^2}\zeta^2(x)\Big)\,d\sigma.
\end{aligned} \tag{7.21}$$

Since $\int_{\partial B_R}\zeta(x)\,d\sigma = 0$, it follows from the Poincaré inequality on ∂B_R that

$$\int\limits_{\partial B_R}\Big(|\nabla_{\tan}\zeta(x)|^2 - \frac{n-1}{R^2}\zeta^2(x)\Big)\,d\sigma \geq 0. \tag{7.22}$$

Now we claim that

$$\frac{R}{n^2}\int\limits_{\partial B_R}\zeta^2(x)\,d\sigma - \frac{1}{n}\int\limits_{\partial B_R} v(x)\zeta(x)\,d\sigma \geq 0. \tag{7.23}$$

To see this, notice that, by (7.8), $\int_{\partial B_R} v(x)\,d\sigma = 0$. Recall that the first Stekloff eigenvalue on B_R is $\frac{1}{R}$, which implies that

$$\int\limits_{\partial B_R} v^2(x)\,d\sigma \leq R\int\limits_{B_R}|\nabla v(x)|^2\,dx.$$

Applying equation (7.5) for v, we have

$$\int_{B_R} |\nabla v(x)|^2 \, dx = \int_{\partial B_R} \frac{\partial v(x)}{\partial \nu} v(x) \, d\sigma = \frac{1}{n} \int_{\partial B_R} \zeta(x) v(x) \, d\sigma \leq \frac{1}{n} \Big(\int_{\partial B_R} v^2(x) \, d\sigma \Big)^{\frac{1}{2}} \Big(\int_{\partial B_R} \zeta^2(x) \, d\sigma \Big)^{\frac{1}{2}}$$
$$\leq \frac{R^{\frac{1}{2}}}{n} \Big(\int_{B_R} |\nabla v(x)|^2 \, d\sigma \Big)^{\frac{1}{2}} \Big(\int_{\partial B_R} \zeta^2(x) \, d\sigma \Big)^{\frac{1}{2}}.$$

This implies

$$\frac{1}{n} \int_{\partial B_R} \zeta(x) v(x) \, d\sigma = \int_{B_R} |\nabla v(x)|^2 \, dx \leq \frac{R}{n^2} \int_{\partial B_R} \zeta^2(x) \, d\sigma.$$

Hence (7.23) holds. Putting (7.22) and (7.23) into (7.21), we conclude that

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{J}_m(u(t), \Omega(t)) \geq 0.$$

This completes the proof. □

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