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Weak solutions of non-isothermal nematic liquid crystal flow in dimension three

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Abstract
For any smooth domain $\Omega \subset \mathbb{R}^3$, we establish the existence of a global weak solution $(\mathbf{u}, \mathbf{d}, \theta)$ to the simplified, non-isothermal Ericksen–Leslie system modeling the hydrodynamic motion of nematic liquid crystals with variable temperature for any initial and boundary data $(\mathbf{u}_0, \mathbf{d}_0, \theta_0) \in \mathbf{H} \times H^1(\Omega, \mathbb{S}^2) \times L^1(\Omega)$, with $\mathbf{d}_0(\Omega) \subset \mathbb{S}^2_+$ (the upper half sphere) and ess $\inf_\Omega \theta_0 > 0$.

Keywords Non-isothermal nematic liquid crystals · Ginzburg–Landau approximation · Entropy inequalities

Mathematics Subject Classification 35A05 · 76A10 · 76D03

1 Introduction

The liquid crystal constitutes a state of matter which is intermediate between the solid and the liquid. In the nematic phase, molecules move like those in fluid, while they tend to reveal preferable orientations. A non-isothermal liquid crystal flow in the nematic phase can be described in terms of three physical variables: the velocity field $\mathbf{u}$ of the underlying fluid, the director field $\mathbf{d}$ representing the averaged orientation of liquid crystal molecules, and the background temperature θ. The

Dedicated to Professor M. Chipot on the occasion of his 70th birthday.

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evolution of the velocity field is governed by the incompressible Navier–Stokes system with stress tensors representing viscous and elastic effects. In the nematic case, the director field is driven by transported negative gradient flow of the Oseen–Frank energy functional which represents the internal microscopic damping [3, 8]. We consider the non-isothermal setting in which the temperature is neither spatial nor temporal homogeneous and thus contributes to total dissipation of the whole system.

A great deal of mathematical theories has been devoted to the study of nematic liquid crystals in the continuum formulation. In pioneering papers [4, 5, 13] Ericksen and Leslie have put forward a PDE model based on the principle of conservation laws and momentum balance. There has been extensive mathematical study of analytic issues of the simplified Ericksen–Leslie system. In 1989 Lin [15] first proposed a simplified Ericksen–Leslie model with one constant approximation for the Oseen–Frank energy: $(\mathbf{u}, \mathbf{d}) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^n \times \mathbb{S}^2$ solves

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \mu \Delta \mathbf{u} - \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}), \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} = \Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3), $P : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ denotes the pressure, $\mu > 0$ represents the viscosity constant of the fluid, and $(\nabla \mathbf{d} \odot \nabla \mathbf{d})_{ij} = \sum_{k=1}^{3} \partial_{x_i} \mathbf{d}^k \partial_{x_j} \mathbf{d}^k$ denotes the Ericksen stress tensor. It is a system of the forced Navier–Stokes equation coupled with the transported harmonic map heat flow to $\mathbb{S}^2$. The readers can consult [25] on the study of the Navier–Stokes equations and [22] for some recent developments on harmonic map heat flow. The rigorous mathematical analysis was initiated by Lin-Liu [17, 18] in which they established the well-posedness of so-called Ginzburg–Landau approximation of (1.1): $(\mathbf{u}, \mathbf{d}) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^n \times \mathbb{R}^3$ satisfies

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \mu \Delta \mathbf{u} - \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}), \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} = \Delta \mathbf{d} + \frac{1}{\varepsilon^2} \left(1 - |\mathbf{d}|^2\right) \mathbf{d}, \end{cases} \tag{1.2}$$

where $\varepsilon > 0$ is the parameter of approximation. They have obtained the existence of a unique, global strong solution in dimension 2 and in dimension 3 under large viscosity μ. They have also studied the existence of suitable weak solutions and their partial regularity in dimension 3, which is analogous to the celebrated regularity theorem by Caffarelli et al. [1] (see also [16]) for the dimension 3 incompressible Navier–Stokes equation. Later on Linet al. [19] adopted a different approach to construct global Leray–Hopf type weak solutions (see [12]) for dimension 2 to (1.1) via the method of small energy regularity estimate. Huang et al. [10] extended the works of [19] to the general Ericksen–Leslie system by a blow up argument.

The existence of global weak solution to (1.1) in dimension three is highly non-trivial due to the appearance of the super-critical nonlinear elastic stress term $\nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d})$. Some preliminary progress was made by Lin and Wang [21], where under the assumption that an initial configuration $\mathbf{d}_0$ lies in the upper half sphere, i.e.,

$$\mathbf{d}_0(\Omega) \subset \mathbb{S}^2_+ := \left\{ y = (y^1, y^2, y^3) \in \mathbb{R}^3 \,:\, |y| = 1,\ y^3 \geq 0 \right\}. \tag{1.3}$$

the existence of global weak solution was constructed by the Ginzburg–Laudau approximation method and a delicate blow-up analysis. See [20] for a review of recent progresses on the mathematical analysis of Ericksen–Leslie system.

Recently there has been considerable interest in the mathematical study for the hydrodynamics of non-isothermal nematic liquid crystals. Recall that a simplified, non-isothermal version of (1.2) can be described as follows. Let $(\mathbf{u}, \mathbf{d}, \theta) \,:\, \Omega \times \mathbb{R}_+ \to \mathbb{R}^n \times \mathbb{R}^3 \times \mathbb{R}_+$ solve

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \nabla \cdot (\mu(\theta) \nabla \mathbf{u}) - \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}), \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} = \Delta \mathbf{d} + \frac{1}{\varepsilon^2}\left(1 - |\mathbf{d}|^2\right)\mathbf{d}, \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta = -\nabla \cdot \mathbf{q} + \mu(\theta)|\nabla \mathbf{u}|^2 + \left|\Delta \mathbf{d} + \frac{1}{\varepsilon^2}(1 - |\mathbf{d}|^2)\mathbf{d}\right|^2, \end{cases} \tag{1.4}$$

where $\mathbf{q} \,:\, \Omega \times \mathbb{R}_+ \to \mathbb{R}^n$ is the heat flux. Feireisl et al. [7] proved the existence of a global weak solution to (1.4) in dimension 3. Correspondingly, non-isothermal version of (1.1) reads $(\mathbf{u}, \mathbf{d}, \theta) \,:\, \Omega \times \mathbb{R}_+ \to \mathbb{R}^n \times \mathbb{S}^2 \times \mathbb{R}_+$ solves

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \nabla \cdot (\mu(\theta) \nabla \mathbf{u}) - \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}), \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} = \Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}, \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta = -\nabla \cdot \mathbf{q} + \mu(\theta)|\nabla \mathbf{u}|^2 + \left|\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}\right|^2. \end{cases} \tag{1.5}$$

Hieber and Prüss [9] have established the existence of a unique local $L^p - L^q$ strong solution to (1.5), which can be extended to a global strong solution provided the initial data is close to an equilibrium state. For the general non-isothermal Ericksen–Leslie system, De Anna and Liu [2] have obtained the existence of global strong solution in Besov spaces provided the Besov norm of the initial data is sufficiently small. On $\mathbb{T}^2$, Li and Xin [14] have showed that there exists a global weak solution to (1.5). A natural question is that in dimension 3 whether (1.5) admits a global weak solution. The main goal of this paper is to give a positive answer under the additional assumption (1.3).

This paper is organized as follows. We devote Sect. 2 to the derivation of thermodynamic consistency of a simplified, non-isothermal Ericksen–Leslie system for nematic liquid crystals. The weak formulation for (1.5) model is demonstrated in Sect. 3. In Sect. 4 we will establish the weak maximum principle for the free drifted Ginzburg–Landau heat flow with homogeneous Neumann boundary condition. In Sect. 5, we will establish a priori estimates and the existence of weak solutions to the non-isothermal Ericksen–Leslie system. In Sect. 6, we will show the existence of weak solutions to the non-isothermal Ericksen-Leslie system through detailed analysis of convergence procedure.

2 Thermal consistency of the non-isothermal nematic models

2.1 Non-isothermal Ginzburg–Landau approximation

First we recall the equations of **u** and **d** in the non-isothermal Ginzburg–Laudau approximation (1.4):

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \operatorname{div}(\mu(\theta)\nabla \mathbf{u} - \nabla \mathbf{d} \odot \nabla \mathbf{d}), \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} = \Delta \mathbf{d} - \mathbf{f}_\varepsilon(\mathbf{d}), \end{cases} \tag{2.1}$$

where $\mathbf{f}_\varepsilon(\mathbf{d}) = \partial_{\mathbf{d}} F_\varepsilon(\mathbf{d})$, $F_\varepsilon(\mathbf{d}) = \frac{(|\mathbf{d}|^2-1)^2}{4\varepsilon^2}$.

The difference between (2.1) and the isothermal case (1.2) is that the viscosity coefficient μ is a function of temperature θ. Here the temperature plays a role as parameters both in the material coefficients and the heat conductivity coefficients, which is to be discussed later. To make the system (2.1) a close system, we need the evolution equation for θ. The equation of thermal dissipation is derived according to *First and Second laws of thermodynamics* [24].

First we introduce some basic concepts in thermodynamics. The internal energy density reads

$$e_\varepsilon^{int} = \frac{1}{2}|\nabla \mathbf{d}|^2 + F_\varepsilon(\mathbf{d}) + \theta,$$

and the Helmholtz free energy is given by

$$\psi_\varepsilon = \frac{1}{2}|\nabla \mathbf{d}|^2 + F_\varepsilon(\mathbf{d}) - \theta \ln \theta.$$

Denote the entropy by η in the *Second law of thermodynamics*, which is determined by temperature through the Maxwell relation

$$\eta = -\frac{\partial \psi_\varepsilon}{\partial \theta} = 1 + \ln \theta. \tag{2.2}$$

The internal energy can be obtained by (negative) Legendre transformation of free energy with respect to η, i.e.,

$$e_\varepsilon^{int} = \psi_\varepsilon + \eta\theta.$$

The heat flux $\mathbf{q}$ in the equations of both θ of (1.4) and (1.5) satisfies the generalized Fourier law:

$$\mathbf{q}(\theta) = -k(\theta)\nabla\theta - h(\theta)(\nabla\theta \cdot \mathbf{d})\mathbf{d} \tag{2.3}$$

where $k(\theta)$ and $h(\theta)$ represent thermal conductivities. The evolution of entropy can be written as follows.

$$\eta_t + \mathbf{u} \cdot \nabla\eta = -\nabla \cdot \mathbf{g} + \Delta_\varepsilon, \tag{2.4}$$

where $\mathbf{g}$ is the entropy flux which is determined by the heat flux through the Clausius–Duhem relation

$$\mathbf{q} = \theta\mathbf{g}, \tag{2.5}$$

and the entropy production $\Delta_\varepsilon \geq 0$ is given by (2.8) below.

The thermal consistency of (1.4) is given by the following proposition.

Proposition 2.1 *Suppose* $(\mathbf{u}, \mathbf{d}, \theta)$ *is a strong solution to* (1.4). *Then*

(1) (*First law of thermodynamics*). *The total energy* $e_\varepsilon^{total} = \frac{1}{2}|\mathbf{u}|^2 + e_\varepsilon^{int}$ *is conservative. More precisely, we have*

$$\frac{D}{Dt}e_\varepsilon^{total} + \nabla \cdot (\Sigma + \mathbf{q}) = 0, \tag{2.6}$$

where

$$\Sigma = P\mathbf{u} - \mu(\theta)\mathbf{u} \cdot \nabla\mathbf{u} + \nabla\mathbf{d} \odot \nabla\mathbf{d} \cdot \mathbf{u} - (\nabla\mathbf{d})^T \frac{D\mathbf{d}}{Dt}, \tag{2.7}$$

and $\frac{D}{Dt} := \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$ *denotes the material derivative.*

(2) (*Second law of thermodynamics*). *The entropy cannot decrease during any irreversible process, which means the entropy production* Δ_ε *is alway non-negative, i.e.*,

$$\Delta_\varepsilon = \frac{1}{\theta}\left(\mu(\theta)|\nabla\mathbf{u}|^2 + \left|\Delta\mathbf{d} + \frac{1}{\varepsilon^2}(1 - |\mathbf{d}|^2)\mathbf{d}\right|^2 - \mathbf{q} \cdot \nabla\theta\right) \geq 0. \tag{2.8}$$

Proof We first prove (2.6). By direct calculations, we have

$$
\begin{aligned}
\frac{D}{Dt}e_\varepsilon^{total} &= \mathbf{u}\cdot\frac{D\mathbf{u}}{Dt} + \nabla\mathbf{d} : \frac{D}{Dt}\nabla\mathbf{d} + f_\varepsilon(\mathbf{d})\cdot\frac{D\mathbf{d}}{Dt} + \frac{D\theta}{Dt} \\
&= \mathbf{u}\cdot \mathrm{div}(-PI + \mu(\theta)\nabla\mathbf{u} - \nabla\mathbf{d}\odot\nabla\mathbf{d}) + \nabla\mathbf{d} : \nabla\frac{D\mathbf{d}}{Dt} - \nabla\mathbf{d}\odot\nabla\mathbf{d} : \nabla\mathbf{u} \\
&\quad + \mathbf{f}_\varepsilon(\mathbf{d})\cdot\frac{D\mathbf{d}}{Dt} - \nabla\cdot\mathbf{q} + \mu(\theta)|\nabla\mathbf{u}|^2 + \left|\Delta\mathbf{d} + \frac{1}{\varepsilon^2}(1-|\mathbf{d}|^2)\mathbf{d}\right|^2 \\
&= \mathrm{div}(-P\mathbf{u} + \mu(\theta)\mathbf{u}\cdot\nabla\mathbf{u} - \nabla\mathbf{d}\odot\nabla\mathbf{d}\cdot\mathbf{u}) - \mu(\theta)|\nabla\mathbf{u}|^2 + \nabla\mathbf{d}\odot\nabla\mathbf{d} : \nabla\mathbf{u} \\
&\quad + \mathrm{div}\Big((\nabla\mathbf{d})^T\frac{D\mathbf{d}}{Dt}\Big) - (\Delta\mathbf{d} - \mathbf{f}_\varepsilon(\mathbf{d}))\cdot\frac{D\mathbf{d}}{Dt} - \nabla\mathbf{d}\odot\nabla\mathbf{d} : \nabla\mathbf{u} - \nabla\cdot\mathbf{q} \\
&\quad + \mu(\theta)|\nabla\mathbf{u}|^2 + \left|\Delta\mathbf{d} + \frac{1}{\varepsilon^2}(1-|\mathbf{d}|^2)\mathbf{d}\right|^2 \\
&= \mathrm{div}\Big(-P\mathbf{u} + \mu(\theta)\mathbf{u}\cdot\nabla\mathbf{u} - \nabla\mathbf{d}\odot\nabla\mathbf{d}\cdot\mathbf{u} + (\nabla\mathbf{d})^T\frac{D\mathbf{d}}{Dt}\Big) - \nabla\cdot\mathbf{q} \\
&= -\mathrm{div}(\Sigma + \mathbf{q}).
\end{aligned}
\tag{2.9}
$$

Note that (2.8) follows directly from (2.2), (2.4), $(1.4)_4$, and (2.3), i.e.

$$
\begin{aligned}
\Delta_\varepsilon &= \frac{1}{\theta}\Big(\mu(\theta)|\nabla\mathbf{u}|^2 + \big|\Delta\mathbf{d} - f_\varepsilon(\mathbf{d})\big|^2 - \mathbf{q}\cdot\nabla\theta\Big) \\
&= \frac{1}{\theta}\Big(\mu(\theta)|\nabla\mathbf{u}|^2 + \big|\Delta\mathbf{d} - f_\varepsilon(\mathbf{d})\big|^2 + k(\theta)|\nabla\theta|^2 + h(\theta)|\nabla\theta\cdot\mathbf{d}|^2\Big) \geq 0.
\end{aligned}
$$

This completes the proof. □

2.2 Non-isothermal simplified Ericksen–Leslie system

As ε tends to 0, due to the penalization effect of $F_\varepsilon(\mathbf{d})$, formally the equation of $\mathbf{d}$ in (2.1) converges to

$$
\partial_t\mathbf{d} + \mathbf{u}\cdot\nabla\mathbf{d} = \Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d},
$$

where $|\mathbf{d}| = 1$. This is a “transported gradient flow” of the Dirichlet energy $\frac{1}{2}\int_\Omega |\nabla\mathbf{d}|^2\,dx$ for maps $\mathbf{d} : \Omega \to \mathbb{S}^2$.

As in the previous section, we introduce the total energy for (1.5):

$$
e^{total} = \frac{1}{2}(|\mathbf{u}|^2 + |\nabla\mathbf{d}|^2) + \theta,
$$

and the entropy evolution equation:

$$
\eta_t + \mathbf{u}\cdot\nabla\eta = -\nabla\cdot\mathbf{g} + \Delta_0, \tag{2.10}
$$

where Δ_0 is the entropy production given by (2.12) below.

The thermal consistency of (1.5) is described by the following proposition.

Proposition 2.2 *Suppose* $(\mathbf{u}, \mathbf{d}, \theta)$ *is a strong solution to* (1.5). *Then*

(1) (*First law of thermodynamics*). *The total energy is conservative, i.e.,*

$$\frac{D}{Dt}e^{total} + \nabla\cdot(\Sigma + \mathbf{q}) = 0, \tag{2.11}$$

where $\Sigma = P\mathbf{u} - \mu(\theta)\mathbf{u}\cdot\nabla\mathbf{u} + \nabla\mathbf{d}\odot\nabla\mathbf{d}\cdot\mathbf{u} - (\nabla\mathbf{d})^T\frac{D\mathbf{d}}{Dt}$.

(2) (*Second law of thermodynamics*). *The entropy production* Δ_0 *is non-negative, i.e.,*

$$\Delta_0 = \frac{1}{\theta}\big(\mu(\theta)|\nabla u|^2 + |\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2 - \mathbf{q}\cdot\nabla\theta\big) \geq 0. \tag{2.12}$$

Proof From (1.5), we can compute

$$\begin{aligned}
\frac{De^{total}}{Dt} &= \frac{D}{Dt}\Big(\frac{1}{2}(|\mathbf{u}|^2 + |\nabla\mathbf{d}|^2) + \theta\Big)\\
&= \mathbf{u}\cdot\frac{D\mathbf{u}}{Dt} + \nabla\mathbf{d} : \frac{D}{Dt}\nabla\mathbf{d} + \frac{D\theta}{Dt}\\
&= \mathbf{u}\cdot\mathrm{div}(-PI + \mu(\theta)\nabla\mathbf{u} - \nabla\mathbf{d}\odot\nabla\mathbf{d})\\
&\quad + \nabla\mathbf{d} : \nabla\frac{D\mathbf{d}}{Dt} - \nabla\mathbf{d}\odot\nabla\mathbf{d} : \nabla\mathbf{u} - \nabla\cdot\mathbf{q} + \mu(\theta)|\nabla\mathbf{u}|^2 + \Big|\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}\Big|^2\\
&= \mathrm{div}(-P\mathbf{u} + \mu(\theta)\mathbf{u}\cdot\nabla\mathbf{u} - \nabla\odot\nabla\mathbf{d}\cdot\mathbf{u}) - \mu(\theta)|\nabla\mathbf{u}|^2 + \nabla\mathbf{d}\odot\nabla\mathbf{d} : \nabla\mathbf{u}\\
&\quad + \mathrm{div}\Big((\nabla\mathbf{d})^T\frac{D\mathbf{d}}{Dt}\Big) - (\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d})\cdot\Delta\mathbf{d} - \nabla\mathbf{d}\odot\nabla\mathbf{d} : \nabla\mathbf{u}\\
&\quad - \mathrm{div}\mathbf{q} + \mu(\theta)|\nabla\mathbf{u}|^2 + |\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2\\
&= -\mathrm{div}(\Sigma + \mathbf{q}),
\end{aligned}$$

where we have used the fact $|\mathbf{d}| = 1$ so that

$$(\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d})\cdot\Delta\mathbf{d} = |\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2.$$

This implies (2.11). From the entropy equation (2.10), Clausius–Duhem's relation (2.5), the temperature equation in (1.5), and (2.3), we can show

$$\begin{aligned}
\Delta_0 =& \frac{1}{\theta}\Big(\mu(\theta)|\nabla\mathbf{u}|^2 + \Big|\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}\Big|^2 - \mathbf{q}\cdot\nabla\theta\Big)\\
=& \frac{1}{\theta}\Big(\mu(\theta)|\nabla\mathbf{u}|^2 + \Big|\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}\Big|^2 + k(\theta)|\nabla\theta|^2 + h(\theta)|\nabla\theta\cdot\mathbf{d}|^2\Big) \geq 0.
\end{aligned}$$

This yields (2.12). □

3 Weak formulation for Ericksen–Leslie system (1.5)

Throughout this paper, we will assume that μ is a continuous function, and h, k are Lipschitz continuous functions, and

$$0 < \underline{\mu} \le \mu(\theta) \le \overline{\mu}, \quad 0 < \underline{k} \le k(\theta), h(\theta) \le \overline{k} \quad \text{for all } \theta > 0, \tag{3.1}$$

where $\underline{\mu}$, $\overline{\mu}$, $\underline{k}$, and $\overline{k}$ are positive constants. We will impose the homogeneous boundary condition for $\mathbf{u}$:

$$\mathbf{u}\Big|_{\partial\Omega} = 0, \quad \frac{\partial \mathbf{d}}{\partial \nu}\Big|_{\partial\Omega} = 0, \tag{3.2}$$

where ν is the outward unit normal vector field of $\partial\Omega$. It is readily seen that (3.2) implies that for Σ given by (2.7), it holds

$$\Sigma \cdot \nu|_{\partial\Omega} = 0. \tag{3.3}$$

We will also impose the non-flux boundary condition for the temperature function so that the heat flux $\mathbf{q}$ satisfies

$$\mathbf{q} \cdot \nu|_{\partial\Omega} = 0. \tag{3.4}$$

Set

$$\begin{aligned} \mathbf{H} = {} & \text{Closure of } C_0^\infty(\Omega;\mathbb{R}^3) \cap \{v : \nabla \cdot v = 0\} \text{ in } L^2(\Omega;\mathbb{R}^3), \\ \mathbf{J} = {} & \text{Closure of } C_0^\infty(\Omega;\mathbb{R}^3) \cap \{v : \nabla \cdot v = 0\} \text{ in } H^1(\Omega;\mathbb{R}^3), \end{aligned}$$

and

$$H^1(\Omega, \mathbb{S}^2) = \big\{\mathbf{d} \in H^1(\Omega, \mathbb{R}^3) : \mathbf{d}(x) \in \mathbb{S}^2 \ a.e.\ x \in \Omega\big\}.$$

There is some difference between the weak formulation of non-isothermal systems (1.4) or (1.5) and that of the isothermal system (1.2) or (1.1). For example, an important feature of a weak solution to (1.2) is the law of energy dissipation

$$\frac{d}{dt}\int_\Omega \big(|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2\big)dx = -2\int_\Omega \big(\mu|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d} - f_\varepsilon(\mathbf{d})|^2\big)dx \le 0, \tag{3.5}$$

or

$$\frac{d}{dt}\int_\Omega \big(|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2\big)dx = -2\int_\Omega \big(\mu|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d} + |\nabla \mathbf{d}|^2\mathbf{d}|^2\big)dx \le 0 \tag{3.6}$$

for (1.1).

In contrast with (3.5) and (3.6), we need to include a weak formulation both the *first law of thermodynamics* (2.11) and the *second law of thermodynamics* (2.12) into (1.4) or(1.5). Namely, the entropy inequality for the temperature equation in (1.4):

$$\begin{aligned} & \partial_t H(\theta) + \mathbf{u} \cdot \nabla H(\theta) \\ & \quad \ge -\mathrm{div}(H'(\theta)\mathbf{q}) + H'(\theta)\big(\mu(\theta)|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d} - f_\varepsilon(\mathbf{d})|^2\big) + H''(\theta)\mathbf{q} \cdot \nabla\theta, \end{aligned} \tag{3.7}$$

or in (1.5):

$$\partial_t H(\theta) + \mathbf{u} \cdot \nabla H(\theta) \geq -\mathrm{div}(H'(\theta)\mathbf{q}) + H'(\theta)\big(\mu(\theta)|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}|^2\big) + H''(\theta)\mathbf{q} \cdot \nabla \theta, \tag{3.8}$$

where H is any smooth, non-decreasing and concave function. More precisely, we have the following weak formulation to the non-isothermal system (1.5).

Definition 3.1 For $0 < T < \infty$, a triple $(\mathbf{u}, \mathbf{d}, \theta)$ is a weak solution to (1.5), (3.8) if the following properties hold:

i) $\mathbf{u} \in L^\infty([0,T], \mathbf{H}) \cap L^2([0,T], \mathbf{J}), \mathbf{d} \in L^2([0,T], H^1(\Omega, \mathbb{S}^2)), \theta \in L^\infty([0,T], L^1(\Omega))$.
ii) For any $\varphi \in C_0^\infty(\overline{\Omega} \times [0,T), \mathbb{R}^3)$, with $\nabla \cdot \varphi = 0$ and $\varphi \cdot \nu|_{\partial\Omega} = 0$, $\psi_1 \in C_0^\infty(\overline{\Omega} \times [0,T), \mathbb{R}^3)$, and $\psi_2 \in C^\infty(\bar{\Omega} \times [0,T))$ with $\psi_2 \geq 0$, it holds

$$\int_0^T \int_\Omega \big(\mathbf{u} \cdot \partial_t \varphi + \mathbf{u} \otimes \mathbf{u} : \nabla \varphi\big) = \int_0^T \int_\Omega (\mu(\theta)\nabla \mathbf{u} - \nabla \mathbf{d} \odot \nabla \mathbf{d}) : \nabla \varphi - \int_\Omega \mathbf{u}_0 \cdot \varphi(\cdot, 0), \tag{3.9}$$

$$\int_0^T \int_\Omega (\mathbf{d} \cdot \partial_t \psi_1 + \mathbf{u} \otimes \mathbf{d} : \nabla \psi_1) = \int_0^T \int_\Omega (\nabla \mathbf{d} : \nabla \psi_1 - |\nabla \mathbf{d}|^2 \mathbf{d} \cdot \psi_1) - \int_\Omega \mathbf{d}_0 \cdot \psi_1(\cdot, 0), \tag{3.10}$$

$$\begin{aligned}\int_0^T \int_\Omega H(\theta)\partial_t \psi_2 + \big(H(\theta)\mathbf{u} - H'(\theta)\mathbf{q}\big) \cdot \nabla \psi_2 &\leq -\int_0^T \int_\Omega \big[H'(\theta)\big(\mu(\theta)|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}|^2\big) - H''(\theta)\mathbf{q} \cdot \nabla \theta\big]\psi_2 \\ &\quad - \int_\Omega H(\theta_0)\psi_2(\cdot, 0),\end{aligned} \tag{3.11}$$

for any smooth, non-decreasing and concave function H.
iii) The following the energy inequality (2.11)

$$\int_\Omega \Big(\frac{1}{2}(|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2) + \theta\Big)(\cdot, t) \leq \int_\Omega \Big(\frac{1}{2}(|\mathbf{u}_0| + |\nabla \mathbf{d}_0|^2) + \theta_0\Big) \tag{3.12}$$

holds for a.e. $t \in [0,T)$.
iv) The initial condition $\mathbf{u}(\cdot, 0) = \mathbf{u}_0, \mathbf{d}(\cdot, 0) = \mathbf{d}_0, \theta(\cdot, 0) = \theta_0$ holds in the weak sense.

Now we state our main result of this paper, which is the following existence theorem of global weak solutions to (1.5).

Theorem 3.1 *For any* $T > 0, \mathbf{u}_0 \in \mathbf{H}, \mathbf{d}_0 \in H^1(\Omega, \mathbb{S}^2)$ *and* $\theta_0 \in L^1(\Omega)$, *if* $\mathbf{d}_0(\Omega) \subset \mathbb{S}^2_+$ *and* $\mathrm{ess\,inf}_\Omega \theta_0 > 0$, *then there exists a global weak solution* $(\mathbf{u}, \mathbf{d}, \theta)$ *to* (1.5), (3.8), *subject to the initial condition* $(\mathbf{u}, \mathbf{d}, \theta) = (\mathbf{u}_0, \mathbf{d}_0, \theta_0)$ *and the boundary condition* (3.2) *and* (3.4) *such that*

(1) $\mathbf{u} \in L^\infty_t L^2_x \cap L^2_t H^1_x$,
(2) $\mathbf{d} \in L^\infty_t H^1_x(\Omega, \mathbb{S}^2)$, *and* $\mathbf{d}(x,t) \in \mathbb{S}^2_+$ *a.e. in* $\Omega \times (0,T)$,
(3) $\theta \in L^\infty_t L^1_x \cap L^p_t W^{1,p}_x$ *for* $1 \le p < 5/4$, $\theta \ge \mathrm{ess\,inf}_\Omega \theta_0$ *a.e. in* $\Omega \times (0,T)$.

The proof of Theorem 3.1 is given in the sections below.

4 Maximum principle with homogeneous Neumann boundary conditions

In this section, we will sketch two a priori estimates for a drifted Ginzburg–Landau heat flow under the homogeneous Neumann boundary condition, which is similar to [21] where the Dirichlet boundary condition is considered. More precisely, for $\varepsilon > 0$, we consider

$$\begin{cases} \partial_t \mathbf{d}_\varepsilon + \mathbf{w} \cdot \nabla \mathbf{d}_\varepsilon = \Delta \mathbf{d}_\varepsilon + \frac{1}{\varepsilon^2}\left(1 - |\mathbf{d}_\varepsilon|^2\right)\mathbf{d}_\varepsilon & \text{in } \Omega \times (0,T), \\ \nabla \cdot \mathbf{w} = 0 & \text{in } \Omega \times (0,T), \\ \mathbf{d}_\varepsilon(x,0) = \mathbf{d}_0(x) & \text{on } \Omega, \\ \mathbf{w} = \frac{\partial \mathbf{d}_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{4.1}$$

Then we have

Lemma 4.1 *For* $0 < T \le \infty$, *assume* $\mathbf{w} \in L^2([0,T], \mathbf{J})$ *and* $\mathbf{d}_0 \in H^1(\Omega, \mathbb{S}^2)$. *Suppose* $\mathbf{d}_\varepsilon \in L^2([0,T]; H^1(\Omega, \mathbb{R}^3))$ *solves* (4.1). *Then*

$$|\mathbf{d}_\varepsilon(x,t)| \le 1 \text{ a.e. } (x,t) \in \Omega \times [0,T]. \tag{4.2}$$

Proof Set

$$v^\varepsilon = (|\mathbf{d}_\varepsilon|^2 - 1)_+ = \begin{cases} |\mathbf{d}_\varepsilon|^2 - 1 & \text{if } |\mathbf{d}_\varepsilon| \ge 1, \\ 0 & \text{if } |\mathbf{d}_\varepsilon| < 1. \end{cases}$$

Then v^ε is a weak solution to

$$\begin{cases} \partial_t v^\varepsilon + \mathbf{w} \cdot \nabla v^\varepsilon = \Delta v^\varepsilon - 2\left(|\nabla \mathbf{d}_\varepsilon|^2 + \frac{1}{\varepsilon^2} v^\varepsilon |\mathbf{d}_\varepsilon|^2\right) \le \Delta v^\varepsilon & \text{in } \Omega \times (0,T), \\ \nabla \cdot \mathbf{w} = 0 & \text{in } \Omega \times (0,T), \\ v^\varepsilon(x,0) = 0 & \text{on } \Omega, \\ \mathbf{w} = \frac{\partial v^\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{4.3}$$

Multiplying (4.3)$_1$ by v^ε and integrating it over $\Omega \times [0, \tau]$ for any $0 < \tau \leq T$, we get

$$\int_\Omega |v^\varepsilon(\tau)|^2 + 2\int_0^\tau \int_\Omega |\nabla v^\varepsilon|^2 \leq -\int_0^\tau \int_\Omega \mathbf{w} \cdot \nabla((v^\varepsilon)^2) = 0.$$

Thus $v^\varepsilon = 0$ a.e. in $\Omega \times [0, T]$ and (4.2) holds. □

Lemma 4.2 *For* $0 < T \leq \infty$, *assume* $\mathbf{w} \in L^2([0, T]; \mathbf{J})$ *and* $\mathbf{d}_0 \in H^1(\Omega; \mathbb{S}^2)$, *with* $\mathbf{d}_0(x) \in \mathbb{S}^2_+$ *a.e* $x \in \Omega$. *If* $\mathbf{d}_\varepsilon \in L^2([0, T]; H^1(\Omega; \mathbb{R}^3))$ *solves* (4.1), *then*

$$\mathbf{d}_\varepsilon^3(x, t) \geq 0 \text{ a.e. } (x, t) \in \Omega \times [0, T]. \tag{4.4}$$

Proof Set $\varphi_\varepsilon(x, t) = \max\{-e^{-\frac{t}{\varepsilon^2}} \mathbf{d}^3{}_\varepsilon(x, t), 0\}$. Then

$$\begin{cases} \partial_t \varphi_\varepsilon + \mathbf{w} \cdot \nabla \varphi_\varepsilon - \Delta \varphi_\varepsilon = \alpha_\varepsilon \varphi_\varepsilon, & \text{in } \Omega \times (0, T), \\ \nabla \cdot \mathbf{w} = 0, & \text{in } \Omega \times (0, T), \\ \varphi_\varepsilon(x, 0) = 0, & \text{on } \Omega, \\ \mathbf{w} = \frac{\partial \varphi_\varepsilon}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0, T), \end{cases} \tag{4.5}$$

where

$$\alpha_\varepsilon(x, t) = \frac{1}{\varepsilon^2}(1 - |\mathbf{d}_\varepsilon(x, t)|^2) - \frac{1}{\varepsilon^2} \leq 0 \text{ a.e. in } \Omega \times [0, T].$$

Multiplying (4.5)$_1$ by φ_ε and integrating over $\Omega \times [0, \tau]$ for $0 < \tau \leq T$, we obtain

$$\begin{aligned} \int_\Omega |\varphi_\varepsilon|^2(\tau) + 2\int_0^\tau \int_\Omega |\nabla \varphi_\varepsilon|^2 &= -\int_0^\tau \int_\Omega \mathbf{w} \cdot \nabla(\varphi_\varepsilon^2) + 2\int_0^\tau \int_\Omega \alpha_\varepsilon |\varphi_\varepsilon|^2 \\ &= 2\int_0^\tau \int_\Omega \alpha_\varepsilon |\varphi_\varepsilon|^2 \leq 0. \end{aligned}$$

Thus $\varphi_\varepsilon = 0$ a.e. in $\Omega \times [0, T]$ and (4.4) holds. □

Finally we need the following minimum principle for the temperature which guarantees the positive lower bound of θ.

Lemma 4.3 *For* $0 < T \leq \infty$, *assume* $\mathbf{w} \in L^2(0, T; \mathbf{J})$, $\theta_0 \in L^1(\Omega)$ *with* $\operatorname{ess\,inf}_\Omega \theta_0 > 0$, *and* $\mathbf{d}_\varepsilon \in L^2([0, T]; H^1(\Omega, \mathbb{R}^3))$. *If* $\theta_\varepsilon \in L_t^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega))$ *solves*

$$\begin{cases} \partial_t \theta_\varepsilon + \mathbf{w} \cdot \nabla \theta_\varepsilon = -\nabla \cdot \mathbf{q}_\varepsilon + \mu(\theta_\varepsilon)|\nabla \mathbf{w}|^2 + |\Delta \mathbf{d}_\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon)|^2, & \text{in } \Omega \times (0, T), \\ \nabla \cdot \mathbf{w} = 0, & \text{in } \Omega \times (0, T), \\ \theta_\varepsilon(x, 0) = \theta_0(x), & \text{on } \Omega, \\ \mathbf{w} = \mathbf{q}_\varepsilon \cdot \nu = 0, & \text{on } \partial\Omega \times (0, T), \end{cases} \tag{4.6}$$

where $\mathbf{q}_\varepsilon = -k(\theta_\varepsilon)\nabla\theta_\varepsilon - h(\theta_\varepsilon)(\nabla\theta_\varepsilon \cdot \mathbf{d}_\varepsilon)\mathbf{d}_\varepsilon$, *then*

$$\theta_\varepsilon(x, t) \geq \operatorname{ess\,inf}_\Omega \theta_0 \text{ a.e. in } \Omega \times [0, T]. \tag{4.7}$$

Proof Let $\theta_\varepsilon^- = \max\left\{\operatorname{ess\,inf}_\Omega \theta_0 - \theta_\varepsilon, 0\right\}$. Then by direct computation, (4.6) implies that

$$\begin{cases} \partial_t \theta_\varepsilon^- + \mathbf{w}\cdot\nabla\theta_\varepsilon^- \le -\nabla\cdot\mathbf{q}_\varepsilon^-, & \text{in } \Omega\times(0,T),\\ \nabla\cdot\mathbf{w}=0, & \text{in } \Omega\times(0,T),\\ \theta_\varepsilon^-(x,0)=0, & \text{on } \Omega,\\ \mathbf{w}=\mathbf{q}_\varepsilon^-\cdot\nu=0, & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{4.8}$$

where $\mathbf{q}_\varepsilon^- = -k(\theta_\varepsilon)\nabla\theta_\varepsilon^- - h(\theta_\varepsilon)(\nabla\theta_\varepsilon^-\cdot\mathbf{d}_\varepsilon)\mathbf{d}_\varepsilon$.

Multiplying $(4.8)_1$ by θ_ε^- and integrating over $\Omega\times[0,\tau]$ for $0<\tau\le T$, we obtain

$$\int_\Omega |\theta_\varepsilon^-|^2(\tau) + 2\int_0^\tau\int_\Omega \underline{k}\left(|\nabla\theta_\varepsilon^-|^2 + |\nabla\theta_\varepsilon^-\cdot\mathbf{d}_\varepsilon|^2\right) \le 0.$$

Therefore $\theta_\varepsilon^- = 0$ a.e. in $\Omega\times[0,T]$, which yields (4.7). □

5 Existence of weak solutions to (5.1)

In this section we will sketch the construction of weak solutions to (5.1) by the Faedo–Galerkin method, which is similar to that by [7, 17]. To simplify the presentation, we only consider the case $\varepsilon = 1$ and construct a weak solution of the following system:

$$\begin{cases} \partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla P = \operatorname{div}(\mu(\theta)\nabla\mathbf{u} - \nabla\mathbf{d}\odot\nabla\mathbf{d}),\\ \nabla\cdot\mathbf{u}=0,\\ \partial_t\mathbf{d} + \mathbf{u}\cdot\nabla\mathbf{d} = \Delta\mathbf{d} - \mathbf{f}(\mathbf{d}),\\ \partial_t\theta + \mathbf{u}\cdot\nabla\theta = -\operatorname{div}\mathbf{q} + \mu(\theta)|\nabla\mathbf{u}|^2 + |\Delta\mathbf{d} - \mathbf{f}(\mathbf{d})|^2, \end{cases} \tag{5.1}$$

where $\mathbf{f}(\mathbf{d}) = \partial_{\mathbf{d}}F(\mathbf{d}) = (|\mathbf{d}|^2-1)\mathbf{d}$.

Let $\left\{\varphi_i\right\}_{i=1}^\infty$ be an orthonormal basis of $\mathbf{H}$ formed by eigenfunctions of the Stokes operator on Ω with zero Dirichlet boundary condition, i.e.,

$$\begin{cases} -\Delta\varphi_i + \nabla P_i = \lambda_i\varphi_i & \text{in } \Omega,\\ \nabla\cdot\varphi_i = 0 & \text{in } \Omega,\\ \varphi_i = 0 & \text{on } \partial\Omega, \end{cases}$$

for $i=1,2,\ldots$, and $0<\lambda_1\le\lambda_2\le\cdots\le\lambda_n\le\cdots$, with $\lambda_n\to\infty$.

Let $\mathbb{P}_m : \mathbf{H}\to\mathbf{H}_m = span\left\{\varphi_1,\varphi_2,\ldots,\varphi_m\right\}$ be the orthogonal projection operator. Consider

$$\begin{cases} \partial_t\mathbf{u}_m = \mathbb{P}_m\left[-\mathbf{u}_m\cdot\nabla\mathbf{u}_m + \operatorname{div}\left(\mu(\theta_m)\nabla\mathbf{u}_m - \nabla\mathbf{d}_m\odot\nabla\mathbf{d}_m\right)\right],\\ \mathbf{u}_m(\cdot,t)\in\mathbf{H}_m, \quad \forall t\in[0,T),\\ \mathbf{u}_m(x,0) = \mathbb{P}_m(\mathbf{u}_0)(x), \quad \forall x\in\Omega, \end{cases} \tag{5.2}$$

$$\begin{cases} \partial_t \mathbf{d}_m + \mathbf{u}_m \cdot \nabla \mathbf{d}_m = \Delta \mathbf{d}_m - f(\mathbf{d}_m), \\ \mathbf{d}_m(x,0) = \mathbf{d}_0(x) \ \ \forall x \in \Omega, \\ \frac{\partial \mathbf{d}_m}{\partial \nu} = 0 \quad \text{on} \ \ \partial\Omega, \end{cases} \tag{5.3}$$

$$\begin{cases} \partial_t \theta_m + \mathbf{u}_m \cdot \nabla \theta_m = \operatorname{div}\big(k(\theta_m)\nabla\theta_m + h(\theta_m)(\nabla\theta_m \cdot \mathbf{d}_m)\mathbf{d}_m\big) \\ \qquad\qquad\qquad\qquad + \mu(\theta_m)|\nabla \mathbf{u}_m|^2 + |\Delta \mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2, \\ \theta_m(x,0) = \theta_0(x) \ \ \forall x \in \Omega, \\ \frac{\partial \theta_m}{\partial \nu} = 0 \quad \text{on} \ \ \partial\Omega. \end{cases} \tag{5.4}$$

Since $\mathbf{u}_m(\cdot, t) \in \mathbf{H}_m$, we can write

$$\mathbf{u}_m(x,t) = \sum_{i=1}^{m} g_m^{(i)}(t)\varphi_i(x),$$

so that (5.2) becomes the following system of ODEs:

$$\frac{d}{dt} g_m^{(i)}(t) = A_{jk}^{(i)} g_m^{(j)}(t) g_m^{(k)}(t) + B_{mj}^{(i)}(t) g_m^{(j)}(t) + C_m^{(i)}(t), \tag{5.5}$$

subject to the initial condition

$$g_m^{(i)}(0) = \int_\Omega \langle \mathbf{u}_0, \varphi_i \rangle, \tag{5.6}$$

for $1 \le i \le m$, where

$$\begin{aligned} A_{jk}^{(i)} &= -\int_\Omega \langle \varphi_j \cdot \nabla \varphi_k, \varphi_i \rangle, \\ B_{mj}^{(i)}(t) &= -\int_\Omega \langle \mu(\mathbf{u}_m) \nabla \varphi_j, \nabla \varphi_i \rangle, \\ C_m^{(i)}(t) &= \int_\Omega (\nabla \mathbf{d}_m \odot \nabla \mathbf{d}_m) : \nabla \varphi_i, \end{aligned}$$

for $1 \le j, k \le m$.

For $T_0 > 0$ and $M > 0$ to be chosen later, suppose $\big(g_m^{(1)}, \dots, g_m^{(m)}\big) \in C^1([0, T_0])$ and

$$\sup_{0 \le t \le T_0} \sum_{i=1}^{m} |g_m^{(i)}(t)|^2 \le M^2. \tag{5.7}$$

Since $\partial_t \mathbf{u}_m, \nabla^2 \mathbf{u}_m \in C^0(\Omega \times [0, T_0])$, the standard theory of parabolic equations implies that there exists a strong solution $\mathbf{d}_m$ to (5.3) such that for any $\delta > 0$, $\partial_t \mathbf{d}_m, \nabla^2 \mathbf{d}_m \in L^p(\Omega \times [\delta, T_0])$ for any $1 \le p < \infty$ (see [11]). Next we can solve (5.4) to obtain a nonnegative, strong solution θ_m. In fact, observe that

$$k(\theta_m)\nabla\theta_m + h(\theta_m)(\nabla\theta_m \cdot \mathbf{d}_m)\mathbf{d}_m = D(\theta_m)\nabla\theta_m,$$

where $(D_{ij}(\theta_m)) = (k(\theta_m)\delta_{ij} + h(\theta_m)\mathbf{d}_m^i\mathbf{d}_m^j)$ is uniformly elliptic, and $\mu(\theta_m)|\nabla\mathbf{u}_m|^2 + |\Delta\mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2 \in L^p(\Omega\times[\delta, T_0])$ holds for any $1 < p < \infty$ and $\delta > 0$. Thus by the standard theory of parabolic equations, we can first obtain a unique weak solution θ_m to (5.3) such that $\theta_m \in C^\alpha(\overline{\Omega}\times[\delta, T_0])$ for some $\alpha \in (0, 1)$. This yields that the coefficient matrix $D(\theta_m) \in C(\overline{\Omega}\times[\delta, T_0])$ and hence by the regularity theory of parabolic equations we conclude that $\nabla\theta_m \in L^p(\Omega\times[\delta, T_0])$ for any $1 < p < \infty$ and $\delta > 0$. Now we see that θ_m satisfies

$$\partial_t\theta_m - D_{ij}(\theta_m)\frac{\partial^2\theta_m}{\partial x_i\partial x_j} = D'_{ij}(\theta_m)\frac{\partial\theta_m}{\partial x_i}\frac{\partial\theta_m}{\partial x_j} + \mu(\theta_m)|\nabla\mathbf{u}_m|^2 + |\Delta\mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2,$$

where $|D'_{ij}(\theta_m)| \le |h'(\theta_m)| + |k'(\theta_m)|$ is bounded, since h and k are Lipschitz continuous. Hence by the $W^{2,1}_p$-theory of parabolic equations, $\partial_t\theta_m, \nabla^2\theta_m \in L^p(\Omega\times[\delta, T_0])$ for any $1 < p < \infty$ and $\delta > 0$.

To solve (5.5) and (5.6), we need some apriori estimates. Taking the L^2 inner product of (5.3) with $-\Delta\mathbf{d}_m + \mathbf{f}(\mathbf{d}_m)$ yields

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla\mathbf{d}_m|^2 + 2F(\mathbf{d}_m) &= -2\int_\Omega |\Delta\mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2 + 2\int_\Omega (\mathbf{u}_m\cdot\nabla\mathbf{d}_m)\cdot(\Delta\mathbf{d}_m - \mathbf{f}(\mathbf{d}_m))\\ &\le -\int_\Omega |\Delta\mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2 + \int_\Omega |\mathbf{u}_m\cdot\nabla\mathbf{d}_m|^2, \quad t\in[0, T_0].\end{aligned}$$

It follows from (5.7) that

$$\|\mathbf{u}_m\|_{L^\infty(\Omega\times[0,T_0])} \le M\cdot\max_{1\le i\le m}\|\varphi_i\|_{L^\infty(\Omega)} \le C_m M.$$

Therefore we get

$$\frac{d}{dt}\int_\Omega (|\nabla\mathbf{d}_m|^2 + 2F(\mathbf{d}_m)) + \int_\Omega |\Delta\mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2 \le C_m^2M^2\int_\Omega |\nabla\mathbf{d}_m|^2.$$

This, combined with Gronwall's inequality and $F(\mathbf{d}_0) = 0$, implies

$$\sup_{0\le t\le T_0}\int_\Omega (|\nabla\mathbf{d}_m|^2 + F(\mathbf{d}_m)) + \int_0^{T_0}\int_\Omega |\Delta\mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2 \le e^{C_m^2M^2T_0}\int_\Omega |\nabla\mathbf{d}_0|^2,$$

so that

$$\sup_{0\le t\le T_0}\max_{1\le i,j\le m}\left(|B_{mj}^{(i)}(t)| + |C_m^{(i)}(t)|\right) \le C_0(m, M).$$

Thus we can solve (5.5) and (5.6) to obtain a unique solution $(\tilde{g}_m^{(1)}(t), \dots, \tilde{g}_m^{(m)}(t)) \in C^1([0, T_0])$ such that for all $t \in [0, T_0]$

$$\sum_{i=1}^m |\tilde{g}_m^{(i)}(t)|^2 \le \sum_{i=1}^m |g_m^{(i)}(0)|^2 + C(m, M, \underline{\mu}, \overline{\mu}, \underline{k}, \overline{k})t^2. \tag{5.8}$$

Choose $M = 2 + 2\sum_{i=1}^{m} |g_m^{(i)}(0)|^2$ and $T_0 > 0$ so small that the right-hand side of (5.8) is less than M^2 for all $t \in [0, T_0]$. Set $\tilde{\mathbf{u}}_m : \Omega \times [0, T_0] \to \mathbb{R}^3$ by

$$\tilde{\mathbf{u}}_m(x,t) = \sum_{i=1}^{m} \tilde{g}_m^{(i)}(t)\varphi_i(x).$$

Then $\mathcal{L}(\mathbf{u}_m) = \tilde{\mathbf{u}}_m$ defines a map from $\mathbf{U}(T_0)$ to $\mathbf{U}(T_0)$, where

$$\mathbf{U}(T_0) = \left\{ \mathbf{u}_m(x,t) = \sum_{i=1}^{m} g_m^{(i)}(t)\varphi_i(x) : \max_{t\in[0,T_0]} \sum_{i=1}^{m} |g_m^{(i)}(t)|^2 \le M^2, \quad \mathbf{u}_m(0) = \mathbb{P}_m \mathbf{u}_0 \right\}.$$

Since $\mathbf{U}(T_0)$ is a closed, convex subset of $H_0^1(\Omega)$ and $\mathcal{L}$ is a compact operator, it follows from the Leray–Schauder theorem that $\mathcal{L}$ has a fixed point $\mathbf{u}_m \in \mathbf{U}(T_0)$ for the approximation system (5.2), and a classical solution $\mathbf{d}_m$ to (5.3) and θ_m to (5.4) on $\Omega \times [0, T_0]$, see [6].

Next, we will establish a priori estimates and show that the solution can be extended to [0, T]. To do it, taking the L^2 inner product of (5.2) and (5.3) by $\mathbf{u}_m$ and $-\Delta\mathbf{d}_m + \mathbf{f}(\mathbf{d}_m)$ respectively, and adding together these two equations, we get that for $t \in [0, T_0]$,

$$\frac{d}{dt}\int_\Omega (|\mathbf{u}_m|^2 + |\nabla\mathbf{d}_m|^2 + 2F(\mathbf{d}_m)) + 2\int_\Omega \mu(\theta_m)|\nabla\mathbf{u}_m|^2 + |\Delta\mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2 = 0, \tag{5.9}$$

where we use the identities

$$\int_\Omega \mathbf{u}_m \cdot \operatorname{div}(\nabla\mathbf{d}_m \odot \nabla\mathbf{d}_m) = \int_\Omega (\mathbf{u}_m \cdot \nabla\mathbf{d}_m) \cdot \Delta\mathbf{d}_m,$$
$$\int_\Omega (\mathbf{u}_m \cdot \nabla\mathbf{d}_m) \cdot \mathbf{f}(\mathbf{d}_m) = \int_\Omega \mathbf{u}_m \cdot \nabla F(\mathbf{d}_m) = 0.$$

We can derive from (5.9) that

$$\sup_{0\le t\le T_0} \int_\Omega (|\mathbf{u}_m|^2 + |\nabla\mathbf{d}_m|^2 + 2F(\mathbf{d}_m)) + 2\int_0^{T_0}\int_\Omega \mu(\theta_m)|\nabla\mathbf{u}_m|^2 + |\Delta\mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2$$
$$\le \int_\Omega (|\mathbf{u}_0|^2 + |\nabla\mathbf{d}_0|^2). \tag{5.10}$$

Lemma 4.1 implies that $|\mathbf{d}_m| \le 1$ and $|\mathbf{f}(\mathbf{d}_m)| \le 1$ in $\Omega \times [0, T_0]$, so that

$$\int_0^{T_0}\int_\Omega |\Delta\mathbf{d}_m|^2 \le 2\int_0^{T_0}\int_\Omega (1 + |\Delta\mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2).$$

Hence (5.10) yields thqat

$$\sup_{0\le t\le T_0}\int_\Omega(|\mathbf{u}_m|^2+|\nabla\mathbf{d}_m|^2)+\int_0^{T_0}\int_\Omega(\underline{\mu}|\nabla\mathbf{u}_m|^2+|\Delta\mathbf{d}_m|^2)$$
$$\le\int_\Omega(|\mathbf{u}_0|^2+|\nabla\mathbf{d}_0|^2)+CT_0|\Omega|. \tag{5.11}$$

While the integration of (5.4) over Ω yields

$$\frac{d}{dt}\int_\Omega\theta_m=\int_\Omega(\mu(\theta_m)|\nabla\mathbf{u}_m|^2+|\Delta\mathbf{d}_m-\mathbf{f}(\mathbf{d}_m)|^2). \tag{5.12}$$

Adding (5.9) together with (5.12) and integrating over $[0,T_0]$, we obtain

$$\sup_{0\le t\le T_0}\int_\Omega(|\mathbf{u}_m|^2+|\nabla\mathbf{d}_m|^2+\theta_m)\le\int_\Omega(|\mathbf{u}_0|^2+|\nabla\mathbf{d}_0|^2+\theta_0). \tag{5.13}$$

Next by choosing $H(\theta)=(1+\theta)^\alpha$, $\alpha\in(0,1)$, and multiplying Eq. (5.4) by $H'(\theta_m)=\alpha(1+\theta_m)^{\alpha-1}$, we get

$$\begin{aligned}\partial_t(1+\theta_m)^\alpha+\mathbf{u}_m\cdot\nabla(1+\theta_m)^\alpha\\ =-\mathrm{div}\big(\alpha(1+\theta_m)^{\alpha-1}\mathbf{q}_m\big)+\alpha(1+\theta_m)^{\alpha-1}\big(\mu(\theta_m)|\nabla\mathbf{u}_m|^2+|\Delta\mathbf{d}_m-\mathbf{f}(\mathbf{d}_m)|^2\big)\\ +\alpha(\alpha-1)(1+\theta_m)^{\alpha-2}\mathbf{q}_m\cdot\nabla\theta_m,\end{aligned} \tag{5.14}$$

where $\mathbf{q}_m=-h(\theta_m)\nabla\theta_m-k(\theta_m)(\nabla\theta_m\cdot\mathbf{d}_m)\mathbf{d}_m$.

Integrating (5.14) over $\Omega\times[0,T_0]$ yields

$$\int_0^{T_0}\int_\Omega\alpha(\alpha-1)(1+\theta_m)^{\alpha-2}\mathbf{q}_m\cdot\nabla\theta_m\le\int_{\Omega\times\{T_0\}}(1+\theta_m)^\alpha-\int_\Omega(1+\theta_0)^\alpha. \tag{5.15}$$

Notice that

$$\begin{aligned}\int_0^{T_0}\int_\Omega\alpha(\alpha-1)(1+\theta_m)^{\alpha-2}\mathbf{q}_m\cdot\nabla\theta_m\\ =\alpha(1-\alpha)\int_0^{T_0}\int_\Omega(1+\theta_m)^{\alpha-2}(k(\theta_m)|\nabla\theta_m|^2+h(\theta_m)(\nabla\theta_m\cdot\mathbf{d}_m)^2)\\ \ge\alpha(1-\alpha)\underline{k}\int_0^{T_0}\int_\Omega(1+\theta_m)^{\alpha-2}|\nabla\theta_m|^2\\ \ge\frac{4\alpha(1-\alpha)\underline{k}}{\alpha^2}\int_0^{T_0}\int_\Omega|\nabla\theta_m^{\frac{\alpha}{2}}|^2.\end{aligned}$$

Thus we obtain that

$$\begin{aligned}\int_0^{T_0}\int_\Omega \left|\nabla\theta_m^{\frac{\alpha}{2}}\right|^2 &\le C(\alpha,\underline{k})\int_{\Omega\times\{T_0\}}(1+\theta_m)^\alpha\\ &\le C(\alpha,\underline{k},\Omega)\left(\int_{\Omega\times\{T_0\}}(1+\theta_m)\right)^\alpha\\ &\le C(\alpha,\underline{k},\Omega)\left(1+\int_\Omega(|\mathbf{u}_0|^2+|\nabla\mathbf{d}_0|^2+\theta_0)\right)^\alpha. \end{aligned} \tag{5.16}$$

With (5.13) and (5.16), we can apply an interpolation argument, similar to (4.13) in [7], to conclude that $\theta_m \in L^q(\Omega\times[0,T_0])$ for any $1 \le q < \frac{5}{3}$, and

$$\|\theta_m\|_{L^q(\Omega\times[0,T])} \le C\big(q,\underline{k},\|\mathbf{u}_0\|_{L^2(\Omega)},\|\nabla\mathbf{d}_0\|_{L^2(\Omega)},\|\theta_0\|_{L^1(\Omega)}\big). \tag{5.17}$$

This, together with (5.16) and Hölder's inequality:

$$\int_{\Omega\times[0,T_0]}|\nabla\theta_m|^p \le \left(\int_{\Omega\times[0,T_0]}|\nabla\theta_m|^2\theta_m^{\alpha-2}\right)^{\frac{p}{2}}\left(\int_{\Omega\times[0,T_0]}\theta_m^{(2-\alpha)\frac{p}{2-p}}\right)^{\frac{2-p}{2}},$$

for $\alpha\in(0,1)$ and $1 \le p < 2$, implies that

$$\|\nabla\theta_m\|_{L^p(\Omega\times[0,T_0])} \le C\big(p,\underline{k},\|\mathbf{u}_0\|_{L^2(\Omega)},\|\nabla\mathbf{d}_0\|_{L^2(\Omega)},\|\theta_0\|_{L^1(\Omega)}\big) \tag{5.18}$$

holds for all $p\in[1,5/4)$.

Plugging the estimates (5.11), (5.13), (5.17), and (5.18) into the system (5.2), (5.3), and (5.4), we conclude that

$$\sup_m\left\{\|\partial_t\mathbf{u}_m\|_{L^{\frac{4}{3}}(0,T_0;H^{-1}(\Omega))}+\|\partial_t\mathbf{d}_m\|_{L^{\frac{4}{3}}(0,T_0;L^2(\Omega))}+\|\partial_t\theta_m\|_{L^2(0,T_0;W^{-1,4}(\Omega))}\right\}\le C. \tag{5.19}$$

Therefore, by setting $\big(\mathbf{u}_m(\cdot,T_0),\mathbf{d}_m(\cdot,T_0),\theta_m(\cdot,T_0)\big)$ as then initial data and repeating the same argument, we can extend the solution to the interval $[0,2T_0]$ and eventually obtain a solution $(\mathbf{u}_m,\mathbf{d}_m,\theta_m)$ to the system (5.2), (5.3), (5.4) in $[0,T]$ such that the estimates (5.11), (5.13), (5.17), (5.18), and (5.19) hold with T_0 replaced by T.

The existence of a weak solution to the original system (5.1) will be obtained by passing to the limit of $(\mathbf{u}_m,\mathbf{d}_m,\theta_m)$ as $m\to\infty$. In fact, by Aubin–Lions' compactness lemma [23], we know that there exists $\mathbf{u}\in L_t^\infty L_x^2\cap L_t^2H_x^1(\Omega\times[0,T])$, $\mathbf{d}\in L_t^\infty H_x^1\cap L_t^2H_x^2(\Omega\times[0,T])$, and a nonnegative $\theta\in L_t^\infty L_x^1\cap L_t^pW_x^{1,p}(\Omega\times[0,T])$, for $1<p<\frac{5}{4}$, such that, after passing to a subsequence,

$$\begin{cases}\mathbf{u}_m\to\mathbf{u} & \text{in } L^2(\Omega\times[0,T]),\\ (\mathbf{d}_m,\nabla\mathbf{d}_m)\to(\mathbf{d},\nabla\mathbf{d}) & \text{in } L^2(\Omega\times[0,T]),\\ \theta_m\to\theta & \textit{a.e. and } \text{in } L^{p_1}(\Omega\times[0,T]),\ \forall 1<p_1<\frac{5}{3},\\ \nabla\mathbf{u}_m\rightharpoonup\nabla\mathbf{u} & \text{in } L^2(\Omega\times[0,T]),\\ \nabla^2\mathbf{d}_m\rightharpoonup\nabla^2\mathbf{d} & \text{in } L^2(\Omega\times[0,T]),\\ \nabla\theta_m\rightharpoonup\nabla\theta & \text{in } L^{p_2}(\Omega\times[0,T]),\ \forall 1<p_2<\frac{5}{4}.\end{cases}$$

Since $\mu \in C([0,\infty))$ is bounded, we have that

$$\mu(\theta_m) \to \mu(\theta) \ \text{ in } \ L^p(\Omega \times [0,T]),\ \forall 1 \le p < \infty,$$

and

$$\mu(\theta_m)\nabla \mathbf{u}_m \rightharpoonup \mu(\theta)\nabla \mathbf{u} \ \text{ in } \ L^2(\Omega \times [0,T]).$$

After passing $m \to \infty$ in Eqs. (5.2) and (5.3), we see that $(\mathbf{u},\mathbf{d},\theta)$ satisfies Eqs. $(5.1)_1$, $(5.1)_2$, and $(5.1)_3$ in the weak sense.

Next we want to verify that θ satisfies

$$\begin{aligned} &\int_0^T \int_\Omega \big(H(\theta)\partial_t\psi + (H(\theta)\mathbf{u} - H'(\theta)\mathbf{q})\cdot\nabla\psi\big) \\ &\quad \le -\int_0^T \int_\Omega \big[H'(\theta)(\mu(\theta)|\nabla\mathbf{u}|^2 + |\Delta\mathbf{d} - \mathbf{f}(\mathbf{d})|^2) - H''(\theta)\mathbf{q}\cdot\nabla\theta\big]\psi \\ &\qquad - \int_\Omega H(\theta_0)\psi(\cdot,0) \end{aligned} \tag{5.20}$$

holds for any smooth, non-decreasing and concave function H, and $\psi \in C_0^\infty(\overline{\Omega}\times[0,T))$ with $\psi \ge 0$. Here $\mathbf{q} = -k(\theta)\nabla\theta - h(\theta)(\nabla\theta\cdot\mathbf{d})\mathbf{d}$. Observe that by choosing $H(t) = t$, (5.20) yields that θ solves $(5.1)_4$ in the weak sense, namely,

$$\begin{aligned} &\int_0^T \int_\Omega \big(\theta\partial_t\psi + (\theta\mathbf{u} - \mathbf{q})\cdot\nabla\psi\big) \\ &\quad \le -\int_0^T \int_\Omega (\mu(\theta)|\nabla\mathbf{u}|^2 + |\Delta\mathbf{d} - \mathbf{f}(\mathbf{d})|^2)\psi - \int_\Omega \theta_0\psi(\cdot,0). \end{aligned} \tag{5.21}$$

In order to show (5.20), first observe that multiplying Eq. (5.4) by $H'(\theta_m)\psi$, integrating over $\Omega\times[0,T]$, and employing the regularity of $\theta_m, \mathbf{u}_m, \mathbf{d}_m$ implies

$$\begin{aligned} &\int_0^T \int_\Omega \big(H(\theta_m)\partial_t\psi + (H(\theta_m)\mathbf{u}_m - H'(\theta_m)\mathbf{q}_m)\cdot\nabla\psi\big) \\ &\quad = -\int_0^T \int_\Omega \big[H'(\theta_m)(\mu(\theta_m)|\nabla\mathbf{u}_m|^2 + |\Delta\mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2) - H''(\theta_m)\mathbf{q}_m\cdot\nabla\theta_m\big]\psi \\ &\qquad - \int_\Omega H(\theta_0)\psi(\cdot,0), \end{aligned} \tag{5.22}$$

where $\mathbf{q}_m = -k(\theta_m)\nabla\theta_m - h(\theta_m)(\nabla\theta_m\cdot\mathbf{d}_m)\mathbf{d}_m$.

It follows from Lemma 4.3 that $\theta_m \ge \operatorname{ess\,inf}_\Omega\theta_0$ a.e.. Without loss of generality, we assume $H(0) = 0$ so that $H(\theta_m) \ge H(\operatorname{ess\,inf}_\Omega\theta_0) \ge 0$ since H is nondecreasing. From $H'' \le 0$, we conclude that $0 \le H'(\theta_m) \le H'(\operatorname{ess\,inf}_\Omega\theta_0)$. From the concavity of H, we have

$$\frac{1}{|\Omega|}\int_\Omega H(\theta_m) \le H\left(\frac{1}{|\Omega|}\int_\Omega \theta_m\right)$$

so that

$$\{H(\theta_m)\} \text{ is bounded in } L_t^\infty L_x^1 \cap L_t^p W_x^{1,p}(\Omega\times[0,T]),\ \forall 1<p<\frac{5}{4}.$$

This, combined with the bounds on $\theta_m, \mathbf{u}_m, \mathbf{d}_m$ and (5.22), implies that

$$\int_0^T\int_\Omega H''(\theta_m)\mathbf{q}_m\cdot\nabla\theta_m\psi$$
$$=\int_0^T\int_\Omega (|\sqrt{-H''(\theta_m)k(\theta_m)\psi}\nabla\theta_m|^2 + |\sqrt{-H''(\theta_m)h(\theta_m)\psi}(\nabla\theta_m\cdot\mathbf{d}_m)|^2)$$

is uniformly bounded. For any fixed $l\in\mathbb{N}^+$, since

$$\sqrt{\min\{-H''(\theta_m),l\}k(\theta_m)\psi}\nabla\theta_m \rightharpoonup \sqrt{\min\{-H''(\theta),l\}k(\theta)\psi}\nabla\theta,$$

and

$$\sqrt{\min\{-H''(\theta_m),l\}h(\theta_m)\psi}(\nabla\theta_m\cdot\mathbf{d}_m) \rightharpoonup \sqrt{\min\{-H''(\theta),l\}h(\theta)\psi}(\nabla\theta\cdot\mathbf{d})$$

in $L^p(\Omega\times[0,T]$ for $1<p<\frac{5}{4}$, we have by the lower semicontinuity that

$$\begin{aligned}\int_0^T\int_\Omega \min\{-H''(\theta),l\}\mathbf{q}\cdot\nabla\theta\psi &\le \liminf_{m\to\infty}\int_0^T\int_\Omega \min\{-H''(\theta_m),l\}\mathbf{q}_m\cdot\nabla\theta_m\psi\\ &\le \liminf_{m\to\infty}\int_0^T\int_\Omega -H''(\theta_m)\mathbf{q}_m\cdot\nabla\theta_m\psi.\end{aligned} \tag{5.23}$$

This, after sending $l\to\infty$, yields

$$\int_0^T\int_\Omega -H''(\theta)\mathbf{q}\cdot\nabla\theta\psi \le \liminf_{m\to\infty}\int_0^T\int_\Omega -H''(\theta_m)\mathbf{q}_m\cdot\nabla\theta_m\psi. \tag{5.24}$$

It follows from the lower semicontinuity again that

$$\begin{aligned}&\int_0^T\int_\Omega \left[H'(\theta)(\mu(\theta)|\nabla\mathbf{u}|^2 + |\Delta\mathbf{d}-\mathbf{f}(\mathbf{d})|^2)\psi\right.\\ &\quad\le \liminf_{m\to\infty}\int_0^T\int_\Omega \left[H'(\theta_m)(\mu(\theta_m)|\nabla\mathbf{u}_m|^2 + |\Delta\mathbf{d}_m-\mathbf{f}(\mathbf{d}_m)|^2)\psi\right..\end{aligned} \tag{5.25}$$

On the other hand, since

$$H(\theta_m)\to H(\theta),\ H(\theta_m)\mathbf{u}_m\to H(\theta)\mathbf{u} \text{ in } L^1(\Omega\times[0,T]),$$

and

$$H'(\theta_m)\mathbf{q}_m \rightharpoonup H'(\theta)\mathbf{q} \ \text{ in } \ L^1(\Omega \times [0,T]),$$

we have

$$\begin{aligned}
&\int_0^T \int_\Omega \big(H(\theta)\partial_t \psi + (H(\theta)\mathbf{u} - H'(\theta)\mathbf{q}) \cdot \nabla \psi\big) \\
&\quad = \lim_{m\to\infty} \int_0^T \int_\Omega \big(H(\theta_m)\partial_t \psi + (H(\theta_m)\mathbf{u}_m - H'(\theta_m)\mathbf{q}_m) \cdot \nabla \psi\big).
\end{aligned} \tag{5.26}$$

Therefore (5.20) follows by passing $m \to \infty$ in (5.22) and applying (5.24), (5.25), and (5.26). This completes the construction of a global weak solution to (5.1). □

6 Convergence and existence of global weak solutions of (1.5)

In this section, we will apply Lemma 4.1, Lemma 4.2, and Lemma 4.3 to analyze the convergence of a sequence of weak solutions $(\mathbf{u}_\varepsilon, \mathbf{d}_\varepsilon, \theta_\varepsilon)$ to the Ginzburg–Landau approximate system (1.4) constructed in the previous section, as $\varepsilon \to 0$, and obtain a global weak solution $(\mathbf{u}, \mathbf{d}, \theta)$ to (1.5).

Here we will employ the pre-compactness theorem by Lin and Wang [21] on approximated harmonic maps to show that $\mathbf{d}_\varepsilon \to \mathbf{d}$ in $L^2([0,T], H^1(\Omega))$ as $\varepsilon \to 0$.

Proof of Theorem 3.1 Let $(\mathbf{u}_\varepsilon, \mathbf{d}_\varepsilon, \theta_\varepsilon)$ be the weak solutions to the Ginzburg–Landau approximate system (1.4), under the boundary condition (3.2), (3.4), obtained from Section 5. Then there exist $C_1, C_2 > 0$ depending only on $\mathbf{u}_0$, $\mathbf{d}_0$, and θ_0 such that

$$\begin{aligned}
&\sup_\varepsilon \Big\{ \|\mathbf{u}_\varepsilon\|_{L^\infty_t L^2_x \cap L^2_t H^1_x(\Omega\times[0,T])} + \|\mathbf{d}_\varepsilon\|_{L^\infty_t H^1_x(\Omega\times[0,T])} \Big\} \le C_1, \\
&\sup_\varepsilon \|\theta_\varepsilon\|_{L^\infty_t L^1_x \cap L^p_t W^{1,p}_x(\Omega\times[0,T])} \le C_2(p), \ \forall\, p \in (1, \frac{5}{4}), \\
&\int_{\Omega\times\{t\}} \Big(|\mathbf{u}_\varepsilon|^2 + |\nabla \mathbf{d}_\varepsilon|^2 + \frac{2}{\varepsilon^2}F(\mathbf{d}_\varepsilon)\Big) + 2\int_0^t \int_\Omega \Big(\mu(\theta_\varepsilon)|\nabla \mathbf{u}_\varepsilon|^2 + |\Delta \mathbf{d}_\varepsilon - \frac{1}{\varepsilon^2}\mathbf{f}(\mathbf{d}_\varepsilon)|^2\Big) \\
&\quad \le \int_\Omega (|\mathbf{u}_0|^2 + |\nabla \mathbf{d}_0|^2), \ \forall t \in [0,T],
\end{aligned} \tag{6.1}$$

$$\int_{\Omega\times\{t\}} (|\mathbf{u}_\varepsilon|^2 + |\nabla \mathbf{d}_\varepsilon|^2 + \frac{2}{\varepsilon^2}F(\mathbf{d}_\varepsilon) + \theta_\varepsilon) \le \int_\Omega (|\mathbf{u}_0|^2 + |\nabla \mathbf{d}_0|^2 + \theta_0), \ \forall t \in [0,T], \tag{6.2}$$

and

$$|\mathbf{d}_\varepsilon| \le 1, \ \mathbf{d}_\varepsilon^3 \ge 0, \ \theta_\varepsilon \ge \operatorname{ess\,inf}_\Omega \theta_0, \ \text{ in } \ \Omega \times [0,T]. \tag{6.3}$$

Applying Eq. (1.4), we can further deduce that

$$\sup_{\varepsilon} \left\{ \|\partial_t u_\varepsilon\|_{L^{\frac{4}{3}}([0,T],H^{-1}(\Omega)} + \|\partial_t \mathbf{d}_\varepsilon\|_{L^{\frac{4}{3}}([0,T],L^2(\Omega))} + \|\partial_t \theta_\varepsilon\|_{L^2([0,T],W^{-1,4}(\Omega)} \right\} < C_3. \tag{6.4}$$

Therefore, after passing to a subsequence, there exist $\mathbf{u} \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(\Omega \times [0,T])$, $\mathbf{d} \in L_t^\infty H_x^1(\Omega \times [0,T])$, $\theta \in L_t^\infty L_x^1 \cap L_t^p W_x^{1,p}(\Omega \times [0,T])$ for $1 < p < \frac{5}{4}$ such that

$$\begin{cases} (\mathbf{u}_\varepsilon, \mathbf{d}_\varepsilon) \to (\mathbf{u}, \mathbf{d}) & \text{in } L^2(\Omega \times (0,T)), \\ (\nabla \mathbf{u}_\varepsilon, \nabla \mathbf{d}_\varepsilon) \rightharpoonup (\nabla \mathbf{u}, \nabla \mathbf{d}) & \text{in } L^2(\Omega \times (0,T)) \end{cases} \tag{6.5}$$

as $\varepsilon \to 0$. Since

$$\int_{\Omega\times[0,T]} F(\mathbf{d}) \le \lim_{\varepsilon} \int_{\Omega\times[0,T]} F(\mathbf{d}_\varepsilon) = 0,$$

we conclude that $|\mathbf{d}| = 1$ a.e. in $\Omega \times [0,T]$. Sending $\varepsilon \to 0$ in equations $(1.4)_{2,3}$, we obtain that

$$\nabla \cdot \mathbf{u} = 0 \; a.e. \; in \; \Omega \times [0,T],$$

and

$$(\partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d}) \times \mathbf{d} = \nabla \cdot (\nabla \mathbf{d} \times \mathbf{d}) \;\; \text{weakly} \;\; \text{in } \Omega \times [0,T],$$

which, combined with the fact that $\mathbf{d}$ is $\mathbb{S}^2$-valued, implies that

$$\partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} = \Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d} \;\; \text{weakly} \;\; \text{in } \Omega \times [0,T]. \tag{6.6}$$

Hence (3.10) holds.

'To verify that $\mathbf{u}$ satisfies Eq. $(1.5)_1$, we need to show that $\nabla \mathbf{d}_\varepsilon$ converges to $\nabla \mathbf{d}$ in $L^2_{\text{loc}}(\Omega \times (0,T))$. which makes sense of $\nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d})$. We also need to justify the convergence of temperature equation $(1.5)_4$. For this purpose, we recall some basic notations and theorems in [21] that are needed in the proof.

For any $0 < a \le 2$, L_1 and $L_2 > 0$, denote by $\mathcal{X}(L_1, L_2, a)$ the space that consists of weak solutions $\mathbf{d}_\varepsilon$ of

$$\Delta \mathbf{d}_\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon) = \tau_\varepsilon \text{ in } \Omega$$

such that

(1) $|\mathbf{d}_\varepsilon| \le 1$ and $\mathbf{d}_\varepsilon^3 \ge -1 + a$ for x a.e. in Ω,
(2) $E_\varepsilon(\mathbf{d}_\varepsilon) = \int_\Omega \frac{1}{2} |\nabla \mathbf{d}_\varepsilon|^2 + 3F_\varepsilon(\mathbf{d}_\varepsilon) dx \le L_1$,
(3) $\|\tau_\varepsilon\|_{L^2(\Omega)} \le L_2$.

The following Theorem concerning the H^1 pre-compactness of $\mathcal{X}(L_1, L_2, a)$ was shown by [21]. □

Theorem 6.1 *For any $a \in (0,2]$, $L_1 > 0$ and $L_2 > 0$, the set $\mathcal{X}(L_1, L_2, a)$ is precompact in $H^1_{\text{loc}}(\Omega;\mathbb{R}^3)$. Namely, if $\{\mathbf{d}_\varepsilon\}$ is a sequence of maps in $\mathcal{X}(L_1, L_2, a)$, then there* *exists a map* $\mathbf{d} \in H^1(\Omega;\mathbb{S}^2)$ *such that, after passing to a possible subsequence,* $\mathbf{d}_\varepsilon \to \mathbf{d}$ *in* $H^1_{\mathrm{loc}}(\Omega;\mathbb{R}^3)$.

We also denote by $\mathcal{Y}(L_1, L_2, a)$ the space that consists of $\mathbf{d} \in H^1(\Omega, \mathbb{S}^2)$ that are so-called stationary approximated harmonic maps, more precisely,

$$\begin{cases} \Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d} = \tau \text{ in } \Omega, \\ \int_\Omega (\nabla\mathbf{d} \odot \nabla\mathbf{d}) : \nabla\varphi - \frac{1}{2}|\nabla\mathbf{d}|^2\nabla\cdot\varphi + \langle\tau, \varphi\cdot\nabla\mathbf{d}\rangle = 0, \end{cases} \tag{6.7}$$

for any $\varphi \in C_0^\infty(\Omega;\mathbb{R}^3)$, and

(1) $\mathbf{d}^{(3)}(x) \geq -1 + a$ for x a.e. in Ω,
(2) $E(\mathbf{d}) = \frac{1}{2}\int_\Omega |\nabla\mathbf{d}|^2 dx \leq L_1$,
(3) $\|\tau\|_{L^2(\Omega)} \leq L_2$.

The following H^1 pre-compactness of stationary approximated harmonic maps was also shown by [21].

Theorem 6.2 *For any* $a \in (0, 2]$, $L_1 > 0$ *and* $L_2 > 0$, *the set* $\mathcal{Y}(L_1, L_2, a)$ *is pre-compact in* $H^1_{\mathrm{loc}}(\Omega;\mathbb{S}^2)$. *Namely, if* $\{\mathbf{d}_i\} \subset \mathcal{Y}(L_1, L_2, a)$ *is a sequence of stationary approximated harmonic maps, with tensor fields* $\{\tau_i\}$, *then there exist* $\tau \in L^2(\Omega, \mathbb{R}^3)$ *and a stationary approximated harmonic map* $\mathbf{d} \in \mathcal{Y}(L_1, L_2, a)$, *with tensor field* τ, *namely,*

$$\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d} = \tau \text{ in } \Omega,$$

such that after passing to a possible subsequence, $\mathbf{d}_i \to \mathbf{d}$ *in* $H^1_{\mathrm{loc}}(\Omega, \mathbb{S}^2)$ *and* $\tau_i \rightharpoonup \tau$ *in* $L^2(\Omega;\mathbb{R}^3)$. *Moreover,* $\mathbf{d} \in W^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{S}^2)$.

Now we sketch the proof the compactness of $\nabla\mathbf{d}_\varepsilon$ in $L^2_{loc}(\Omega \times [0, T])$. It follows from Fatou's lemma and (6.1) that

$$\int_0^T \liminf_{\varepsilon\to 0} \int_\Omega |\Delta\mathbf{d}_\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon)|^2 \leq C_0.$$

We decompose [0, T] into the sets of "good time slices" and "bad time slices". For $\Lambda \gg 1$, set

$$\mathcal{G}^T_\Lambda := \left\{ t \in [0, T] : \liminf_{\varepsilon\to 0} \int_\Omega |\Delta\mathbf{d}_\varepsilon - f_\varepsilon(\mathbf{d}_\varepsilon)|^2(t) \leq \Lambda \right\},$$

and

$$\mathcal{B}^T_\Lambda := [0, T] \backslash \mathcal{G}^T_\Lambda = \left\{ t \in [0, T] : \liminf_{\varepsilon\to 0} \int_\Omega |\Delta\mathbf{d}_\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon)|(t) > \Lambda \right\}.$$

From Chebyshev's inequality, we have

$$|\mathcal{B}_\Lambda^T| \le \frac{C_0}{\Lambda}. \tag{6.8}$$

For any $t \in \mathcal{G}_\Lambda^T$, set $\tau_\varepsilon(t) = \big(\Delta \mathbf{d}_\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon)\big)(t)$. Then Lemmas 4.1 and 4.2 imply that $\big\{\mathbf{d}_\varepsilon(t)\big\} \subset \mathcal{X}(C_0, \Lambda, 1)$. Theorem 6.1 then implies that

$$\begin{cases} \mathbf{d}_\varepsilon(t) \to \mathbf{d}(t) & \text{in } H^1_{\rm loc}(\Omega), \\ F_\varepsilon(\mathbf{d}_\varepsilon) \to 0 & \text{in } L^1_{\rm loc}(\Omega), \\ \tau_\varepsilon(t) \rightharpoonup \tau(t) & \text{in } L^2(\Omega). \end{cases}$$

For any $\varphi \in C_0^\infty(\Omega;\mathbb{R}^3)$, multiplying $\tau_\varepsilon(t)$ by $\varphi \cdot \nabla \mathbf{d}_\varepsilon$ and integrating over Ω yields

$$\int_\Omega (\nabla \mathbf{d}_\varepsilon(t) \odot \nabla \mathbf{d}_\varepsilon(t)) : \nabla\varphi - \Big(\frac{1}{2}|\nabla \mathbf{d}_\varepsilon(t)|^2 + F_\varepsilon(\mathbf{d}_\varepsilon(t))\Big)\nabla\cdot\varphi + \langle \tau_\varepsilon(t), \varphi\cdot\nabla \mathbf{d}_\varepsilon(t)\rangle = 0. \tag{6.9}$$

Passing limit $\varepsilon \to 0$ in (6.9), we get

$$\int_\Omega (\nabla \mathbf{d}(t) \odot \nabla \mathbf{d}(t)) : \nabla\varphi - \frac{1}{2}|\nabla \mathbf{d}(t)|^2\nabla\cdot\varphi + \langle \tau(t), \varphi\cdot\nabla \mathbf{d}(t)\rangle = 0.$$

Hence $\mathbf{d}(t) \in \mathcal{Y}(C_0, \Lambda, 1)$ is a stationary approximated harmonic map. Next we want to show that $\mathbf{d}_\varepsilon \to \mathbf{d}$ strongly in $L_t^2 H_x^1$. To see this, we claim that for any compact $K \subset\subset \Omega$,

$$\lim_{\varepsilon\to 0} \int_{K\times\mathcal{G}_\Lambda^T} |\nabla(\mathbf{d}_\varepsilon - \mathbf{d})|^2 = 0. \tag{6.10}$$

For, otherwise, there exist $\delta_0 > 0$, $K \subset\subset \Omega$ and $\varepsilon_i \to 0$ such that

$$\int_{K\times\mathcal{G}_\Lambda^T} |\nabla(\mathbf{d}_{\varepsilon_i} - \mathbf{d})|^2 \ge \delta_0. \tag{6.11}$$

From (6.5), we have

$$\lim_{\varepsilon_i\to 0} \int_{K\times\mathcal{G}_\Lambda^T} |\mathbf{d}_{\varepsilon_i} - \mathbf{d}|^2 = 0. \tag{6.12}$$

By Fubini's theorem, (6.11) and (6.12), there would exist $t_i \in \mathcal{G}_\Lambda^T$ such that

$$\begin{cases} \lim_{\varepsilon_i\to 0}\int_K |\mathbf{d}_{\varepsilon_i}(t_i) - \mathbf{d}(t_i)|^2 = 0, \\ \int_K |\nabla(\mathbf{d}_{\varepsilon_i}(t_i) - \mathbf{d}(t_i))|^2 \ge \frac{2\delta_0}{T}. \end{cases}$$

Thus $\big\{\mathbf{d}_{\varepsilon_i}(t_i)\big\} \subset \mathcal{X}(C_0, \Lambda, 1)$ and $\big\{\mathbf{d}(t_i)\big\} \subset \mathcal{Y}(C_0, \Lambda, 1)$. It follows from Theorems 6.1 and 6.2 that there exist $\mathbf{d}_1, \mathbf{d}_2 \in \mathcal{Y}(C_0, \Lambda, 1)$ such that

$$\mathbf{d}_{\varepsilon_i}(t_i) \to \mathbf{d}_1 \text{ and } \mathbf{d}(t_i) \to \mathbf{d}_2 \text{ strongly in } H^1(\Omega).$$

Therefore we would have

$$\int_K |\nabla(\mathbf{d}_1 - \mathbf{d}_2)|^2 = \lim_{i\to\infty} \int_K |\nabla\big(\mathbf{d}_\varepsilon(t_i) - \mathbf{d}(t_i)\big)|^2 \geq \frac{2\delta_0}{T},$$

and

$$\int_K |\mathbf{d}_1 - \mathbf{d}_2|^2 = \lim_{i\to\infty} \int_K |\mathbf{d}_{\varepsilon_i}(t_i) - \mathbf{d}(t_i)|^2 = 0.$$

This is clearly impossible. Thus the claim is true.

We can also follow the proof of Theorem 6.1 in [21] to conclude that the small energy regularity criteria holds for every $(x,t) \in K \times \mathcal{G}_\Lambda^T$ so that a finite covering argument, together with estimates for Claim 4.5 in [21], yields

$$\lim_{\varepsilon\to 0} \int_{K\times\mathcal{G}_\Lambda^T} F_\varepsilon(\mathbf{d}_\varepsilon) = 0. \tag{6.13}$$

Hence we have that

$$\lim_{\varepsilon\to 0} \left[\|\mathbf{d}_\varepsilon - \mathbf{d}\|^2_{L_t^2 H_x^1(K\times\mathcal{G}_\Lambda^T)} + \int_{K\times\mathcal{G}_\Lambda^T} F_\varepsilon(\mathbf{d}_\varepsilon) \right] = 0.$$

On the other hand, it follows from (6.1) and (6.8) that

$$\begin{aligned} &\|\mathbf{d}_\varepsilon - \mathbf{d}\|^2_{L_t^2 H_x^1(\Omega\times\mathcal{B}_\Lambda^T)} + \int_{\Omega\times\mathcal{B}_\Lambda^T} F_\varepsilon(\mathbf{d}_\varepsilon) \\ &\quad \leq C\left(\sup_{t>0} \int_\Omega (|\mathbf{u}_\varepsilon|^2 + |\nabla\mathbf{d}_\varepsilon|^2 + F_\varepsilon(\mathbf{d}_\varepsilon)) \right) \left|\mathcal{B}_\Lambda^T\right| \leq \frac{C}{\Lambda}. \end{aligned}$$

Therefore, we would arrive at

$$\lim_{\varepsilon\to 0} \left[\|\mathbf{d}_\varepsilon - \mathbf{d}\|^2_{L_t^2 H_x^1(K\times[0,T])} + \int_{K\times[0,T]} F_\varepsilon(\mathbf{d}_\varepsilon) \right] \leq \frac{C}{\Lambda}.$$

Sending $\Lambda \to \infty$ yields that

$$\lim_{\varepsilon\to 0} \left[\|\mathbf{d}_\varepsilon - \mathbf{d}\|^2_{L_t^2 H_x^1(K\times[0,T])} + \int_{K\times[0,T]} F_\varepsilon(\mathbf{d}_\varepsilon) \right] = 0.$$

Therefore we can conclude that $\mathbf{u}$ solves the equation (3.9), provided we can verify that $\mu(\theta_\varepsilon)\nabla\mathbf{u}_\varepsilon \rightharpoonup \mu(\theta)\nabla\mathbf{u}$ weakly in $L^2(\Omega\times[0,T])$, which will be verified below.

Next we turn to the convergence of θ_ε. For $\alpha \in (0,1)$, set $H(\theta_\varepsilon) = (1+\theta_\varepsilon)^\alpha$. Then from (5.14) we have

$$\begin{aligned} &\partial_t(1+\theta_\varepsilon)^\alpha + \mathbf{u}_\varepsilon \cdot \nabla(1+\theta_\varepsilon)^\alpha \\ &\quad \geq -\mathrm{div}\big(\alpha(1+\theta_\varepsilon)^{\alpha-1}\mathbf{q}_\varepsilon\big) + \alpha(1+\theta_\varepsilon)^{\alpha-1}\big(\mu(\theta_\varepsilon)|\nabla\mathbf{u}_\varepsilon|^2 + |\Delta\mathbf{d}_\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon)|^2\big) \\ &\qquad + \alpha(\alpha-1)(1+\theta_\varepsilon)^{\alpha-2}\mathbf{q}_\varepsilon \cdot \nabla\theta_\varepsilon. \end{aligned} \tag{6.14}$$

Integrating (6.14) over $\Omega \times [0, T]$, by the assumption (3.1) on μ, and the bound (6.1) on $\mathbf{u}_\varepsilon$, $\mathbf{d}_\varepsilon$ and θ_ε, we can derive that

$$\sup_{\varepsilon>0}\sup_{0<t<T}\int_\Omega (1+\theta_\varepsilon)^{\alpha-2}|\nabla\theta_\varepsilon|^2 < \infty.$$

Therefore we conclude that $\theta_\varepsilon^{\frac{\alpha}{2}} \in L^2_t H^1_x$ and $\theta_\varepsilon \in L^\infty_t L^1_x$ are uniformly bounded. By interpolation, we would have that for $1 \le p < 5/4$,

$$\sup_{\varepsilon>0}\|\theta_\varepsilon\|_{L^p_t W^{1,p}_x(\Omega\times[0,T])} < \infty.$$

From Eq. $(5.1)_4$, we have that for $1 \le q < \frac{30}{23}$,

$$\begin{aligned}\sup_{\varepsilon>0}\|\partial_t\theta_\varepsilon\|_{L^1_t W^{-1,q}_x} &\le \sup_{\varepsilon>0}\Big(C\|\mathbf{u}_\varepsilon\theta_\varepsilon\|_{L^q_t L^q_x} + C\|\nabla\theta_\varepsilon\|_{L^q_t L^q_x} \\ &\quad + C\Big\||\nabla\mathbf{u}_\varepsilon|^2 + |\Delta\mathbf{d}_\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon)|^2\Big\|_{L^1_t L^1_x}\Big) \\ &\le C\sup_{\varepsilon>0}\Big(\|\mathbf{u}_\varepsilon\|_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}}\|\theta_\varepsilon\|_{L_t^{\frac{10q}{10-3q}} L_x^{\frac{10q}{10-3q}}} + \|\nabla\theta_\varepsilon\|_{L^q_t L^q_x}\Big) + C \\ &< \infty.\end{aligned}$$

Hence, by Aubin–Lions' compactness Lemma [23] again, up to a subsequence, there exists $\theta \in L^\infty_t L^1_x \cap L^p_t W^{1,p}_x$ for $1 \le p < \frac{5}{4}$ such that

$$\begin{cases}\theta_\varepsilon \to \theta & \text{in } L^p(\Omega\times(0,T)),\\ \nabla\theta_\varepsilon \rightharpoonup \nabla\theta & \text{in } L^p(\Omega\times(0,T)),\end{cases}$$

as $\varepsilon \to 0$.

After taking another subsequence, we may assume that $(\mathbf{u}_\varepsilon, \mathbf{d}_\varepsilon, \theta_\varepsilon)$ converge to $(\mathbf{u}, \mathbf{d}, \theta)$ a.e. in $\Omega \times [0, T]$.

Since $\{\mu(\theta_\varepsilon)\}$ is uniformly bounded in $L^\infty(\Omega\times[0,T])$, $\mu(\theta_\varepsilon) \to \mu(\theta)$ a.e. in $\Omega \times [0, T]$ and $\nabla\mathbf{u}_\varepsilon \rightharpoonup \nabla\mathbf{u}$ in $L^2(\Omega\times[0,T])$, it follows that

$$\mu(\theta_\varepsilon)\nabla\mathbf{u}_\varepsilon \rightharpoonup \mu(\theta)\nabla\mathbf{u} \ \text{ in } \ L^2(\Omega\times[0,T]).$$

Thus we verify that (3.9) holds.

Taking the L^2 inner product of $\mathbf{u}_\varepsilon$, $\mathbf{d}_\varepsilon$, θ_ε in (5.1) with respect to $\mathbf{u}_\varepsilon, -\Delta\mathbf{d}_\varepsilon + \mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon), 1$, and adding the resulting equations together, we have the following energy law:

$$\frac{d}{dt}\int_\Omega\Big(\frac{1}{2}|\mathbf{u}_\varepsilon|^2 + \frac{1}{2}|\nabla\mathbf{d}_\varepsilon|^2 + F_\varepsilon(\mathbf{d}_\varepsilon) + \theta_\varepsilon\Big) = 0. \tag{6.15}$$

Taking $\varepsilon \to 0$, this implies that $|\mathbf{d}| = 1$ and

$$\int_\Omega\Big(\frac{1}{2}|\mathbf{u}|^2 + \frac{1}{2}|\nabla\mathbf{d}|^2 + \theta\Big)(t) \le \int_\Omega\Big(\frac{1}{2}|\mathbf{u}_0|^2 + \frac{1}{2}|\nabla\mathbf{d}_0| + \theta_0\Big),\ \forall 0 \le t \le T.$$

Hence the global energy inequality (3.12) holds.

It remains to show that (3.8) follows by passing limit $\varepsilon \to 0$ in (3.7). This can be done exactly as in the last part of the previous section. For any smooth, nondecreasing, concave function H, and $\psi \in C_0^\infty(\overline{\Omega} \times [0,T))$, recall from (5.20) that

$$\begin{aligned}&\int_0^T\int_\Omega \big(H(\theta_\varepsilon)\partial_t\psi + (H(\theta_\varepsilon)\mathbf{u}_\varepsilon - H'(\theta_\varepsilon)\mathbf{q}_\varepsilon)\cdot\nabla\psi\big)\\ &\quad\le -\int_0^T\int_\Omega [H'(\theta_\varepsilon)(\mu(\theta_\varepsilon)|\nabla u_\varepsilon|^2 + |\Delta\mathbf{d}_\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon)|^2) - H''(\theta_\varepsilon)\mathbf{q}_\varepsilon\cdot\nabla\theta_\varepsilon]\psi\\ &\qquad -\int_\Omega H(\theta_0)\psi(\cdot,0).\end{aligned} \tag{6.16}$$

Assume $H(0)=0$. Then the concavity of H, $0 \le H'(\theta_\varepsilon) \le H'(\operatorname{ess\,inf}_\Omega\theta_0)$, and the uniform bound on θ_ε imply that

$$\{H(\theta_\varepsilon)\} \text{ is bounded in } L_t^\infty L_x^1 \cap L_t^p W_x^{1,p}(\Omega\times[0,T]),\ \forall 1<p<\frac{5}{4}.$$

Together with the bounds on $\mathbf{u}_\varepsilon, \mathbf{d}_\varepsilon$, and (6.16), we have that

$$\begin{aligned}&\int_0^T\int_\Omega H''(\theta_\varepsilon)\mathbf{q}_\varepsilon\cdot\nabla\theta_\varepsilon\psi\\ &\quad=\int_0^T\int_\Omega (|\sqrt{-H''(\theta_\varepsilon)k(\theta_\varepsilon)\psi}\nabla\theta_\varepsilon|^2 + |\sqrt{-H''(\theta_\varepsilon)h(\theta_m)\psi}(\nabla\theta_\varepsilon\cdot\mathbf{d}_\varepsilon)|^2)\end{aligned}$$

is uniformly bounded. By an argument similar to (5.24), we can show that

$$\int_0^T\int_\Omega -H''(\theta)\mathbf{q}\cdot\nabla\theta\psi \le \liminf_{\varepsilon\to0}\int_0^T\int_\Omega -H''(\theta_\varepsilon)\mathbf{q}_\varepsilon\cdot\nabla\theta_\varepsilon\psi. \tag{6.17}$$

Observe that

$$\Delta\mathbf{d}_\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon) = \partial_t\mathbf{d}_\varepsilon + \mathbf{u}_\varepsilon\cdot\nabla\mathbf{d}_\varepsilon \rightharpoonup \partial_t\mathbf{d} + \mathbf{u}\cdot\nabla\mathbf{d} = \Delta\mathbf{d} + |\Delta\mathbf{d}|^2\mathbf{d} \quad \text{in} \quad L^2(\Omega\times[0,T]),$$

and $\{H'(\theta_\varepsilon)\}$ is uniformly bounded in $L^\infty(\Omega\times[0,T])$. It follows from the lower semicontinuity that

$$\begin{aligned}&\int_0^T\int_\Omega \big[H'(\theta)(\mu(\theta)|\nabla\mathbf{u}|^2 + |\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2)\psi\\ &\quad\le \liminf_{\varepsilon\to0}\int_0^T\int_\Omega \big[H'(\theta_\varepsilon)(\mu(\theta_\varepsilon)|\nabla\mathbf{u}_\varepsilon|^2 + |\Delta\mathbf{d}_\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon)|^2)\psi.\end{aligned} \tag{6.18}$$

On the other hand, since

$$H(\theta_\varepsilon)\to H(\theta),\ H(\theta_\varepsilon)\mathbf{u}_\varepsilon \to H(\theta)\mathbf{u} \text{ in } L^1(\Omega\times[0,T]),$$

and

$$H'(\theta_\varepsilon)\mathbf{q}_\varepsilon \rightharpoonup H'(\theta)\mathbf{q} \text{ in } L^1(\Omega\times[0,T]),$$

we have

$$\begin{aligned}&\int_0^T\int_\Omega \big(H(\theta)\partial_t\psi + (H(\theta)\mathbf{u} - H'(\theta)\mathbf{q})\cdot\nabla\psi\big)\\ &\quad= \lim_{\varepsilon\to 0}\int_0^T\int_\Omega \big(H(\theta_\varepsilon)\partial_t\psi + (H(\theta_\varepsilon)\mathbf{u}_\varepsilon - H'(\theta_\varepsilon)\mathbf{q}_\varepsilon)\cdot\nabla\psi\big).\end{aligned} \tag{6.19}$$

Therefore (3.11) follows by passing $\varepsilon \to 0$ in (6.16) and applying (6.17), (6.18), and (6.19). This completes the construction of a global weak solution to (1.5). □

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Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interest.

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