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Weak compactness property of simplified nematic liquid crystal flows in dimension two

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Abstract

For any bounded smooth domain $\Omega \subset \mathbb{R}^2$, we will establish the convergence of weak solutions of the Ginzburg-Landau approximation of the simplified Ericksen-Leslie system to a weak solution of the simplified Ericksen-Leslie system associated with either uniaxial or biaxial nematics, as the Ginzburg-Landau parameter tends to zero. We will also show the compactness property of weak solutions to the simplified Ericksen-Leslie system associated with either uniaxial or biaxial nematics. These results follow from the compensated compactness property of the Ericksen stress tensors, which are obtained by the Pohozaev argument for the Ginzburg-Landau approximation of the simplified Ericksen-Leslie system and the L^p-estimate ($1 \leq p < 2$) of the Hopf differential for the simplified Ericksen-Leslie system respectively.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and $\mathcal{N} \subset \mathbb{R}^L$ (for $L \geq 2$) be a smooth compact Riemannian manifold without boundary, and $0 < T \leq \infty$. We will formulate a simplified Ericksen-Leslie system modeling the hydrodynamics of nematic liquid crystals with the orientational director field taking values in $\mathcal{N}$:

$$\begin{cases} u_t + u \cdot \nabla u - \Delta u + \nabla P = -\nabla \cdot \big(\nabla v \odot \nabla v\big), \\ \nabla \cdot u = 0, \\ v_t + u \cdot \nabla v = \Delta v + A(v)(\nabla v, \nabla v), \end{cases} \quad \text{in } Q_T \equiv \Omega \times (0, T), \tag{1.1}$$

where $(u(\mathbf{x}, t), v(\mathbf{x}, t), P(\mathbf{x}, t)) : Q_T \to \mathbb{R}^2 \times \mathcal{N} \times \mathbb{R}$ represents the fluid velocity field, the orientational director field of the nematic liquid crystal material valued in $\mathcal{N}$, and the pressure function respectively, $\big(\nabla v \odot \nabla v\big)_{ij} = \nabla_{\mathbf{x}_i} v \cdot \nabla_{\mathbf{x}_j} v$ for $i, j = 1, 2$ represents the Ericksen-Leslie stress tensor, and $A(y)(\cdot, \cdot)$ is the second fundamental form of $\mathcal{N}$ in $\mathbb{R}^L$ at

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the point $y \in \mathcal{N}$. We remark that the formulation of the third equation of (1.1) arises from the fact that the angular momentum balance law for v obeys the constraint $v(\Omega) \subset \mathcal{N}$, which holds for any spatial dimension of Ω.

This simplified Ericksen-Leslie system (1.1) into $\mathcal{N}$ covers and unifies two important cases in the hydrodynamics of nematic liquid crystals:

(1) If $\mathcal{N} = \mathbb{S}^2$, then the system (1.1) reduces to the simplified Ericksen-Leslie system for uniaxial nematics proposed by [4, 15], and [17]

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla P = -\nabla \cdot \big(\nabla v \odot \nabla v\big), \\ \nabla \cdot u = 0, \\ \partial_t v + u \cdot \nabla v = \Delta v + |\nabla v|^2 v, \end{cases} \quad \text{in } Q_T, \tag{1.2}$$

for $(u(\mathbf{x}, t), v(\mathbf{x}, t), P(\mathbf{x}, t)) : Q_T \to \mathbb{R}^2 \times \mathbb{S}^2 \times \mathbb{R}$. In dimension two, the existence of a unique global weak solution of (1.2), under an initial and boundary condition, has been proved in [22, 25], which satisfies the energy inequality and has at most finitely many singular times, see also [9, 10], and [32]. Very recently, the authors in [14] have constructed an example of finite time singularity. In dimension three, a global weak solution has been constructed in [24] when the initial data $v_0 \in \mathbb{S}^2_+$. Examples of finite time singularity have been constructed by [11]. The reader can consult the survey article [23] and the references therein.

(2) If

$$\mathcal{N} = \Big\{(\mathbf{y}_1, \mathbf{y}_2) \in \mathbb{S}^2 \times \mathbb{S}^2 \;\Big|\; \mathbf{y}_1 \cdot \mathbf{y}_2 = 0\Big\} \subset \mathbb{R}^6,$$

and if we set $v(\mathbf{x}, t) = (n(\mathbf{x}, t), m(\mathbf{x}, t)) : Q_T \to \mathbb{S}^2 \times \mathbb{S}^2$, with $n \cdot m = 0$, then the system (1.1) becomes the following simplified Ericksen-Leslie system for biaxial nematics:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla P = -\nabla \cdot \big(\nabla n \odot \nabla n + \nabla m \odot \nabla m\big), \\ \nabla \cdot u = 0, \\ \partial_t n + u \cdot \nabla n = \Delta n + |\nabla n|^2 n + \langle \nabla n, \nabla m \rangle m, \\ \partial_t m + u \cdot \nabla m = \Delta m + |\nabla m|^2 m + \langle \nabla m, \nabla n \rangle n, \\ n \cdot m = 0, \end{cases} \quad \text{in } Q_T. \tag{1.3}$$

This is a simplified version of the hydrodynamics of biaxial nematics model proposed by E. Grovers and G. Vertogen [6–8] which is based on the Landau-De Gennes Q-tensor theory for nematic liquid crystals [2]. In dimensional two, by extending the techniques developed in [25] the existence of a unique global weak solution to (1.3) has recently been shown in [16], which is smooth off at most finitely many singular times.

A natural approach to construct a weak solution of (1.1), subject to the initial-boundary condition (1.5), is to consider the Ginzburg-Landau approximate system of (1.1) (cf. [18] and [19]). More precisely, for any $\delta > 0$ set the δ-neighborhood of $\mathcal{N}$ by

$$\mathcal{N}_\delta = \Big\{\mathbf{y} \in \mathbb{R}^L \;\Big|\; \mathrm{dist}(\mathbf{y}, \mathcal{N}) < \delta\Big\},$$

where $\mathrm{dist}(\mathbf{y}, \mathcal{N})$ denotes the distance function of $\mathbf{y}$ to $\mathcal{N}$. Let $\Pi_{\mathcal{N}} : \mathcal{N}_\delta \to \mathcal{N}$ denote the nearest point projection map. It is well-known that there exists an $\delta_{\mathcal{N}} = \delta(\mathcal{N}) > 0$ such that both $\mathrm{dist}(\mathbf{y}, \mathcal{N})$ and $\Pi_{\mathcal{N}}(\mathbf{y})$ are smooth for $\mathbf{y} \in \mathcal{N}_{2\delta_{\mathcal{N}}}$. Let $\chi(s) \in C^\infty([0, \infty))$ be a monotone increasing function such that

$$\chi(s) = \begin{cases} s, & \text{if } 0 \le s \le \delta_{\mathcal{N}}^2, \\ 4\delta_{\mathcal{N}}^2, & \text{if } s \ge 4\delta_{\mathcal{N}}^2. \end{cases}$$

Consider the following Ginzburg-Landau energy functional for the director field $v : \Omega \to \mathbb{R}^L$:

$$E_\varepsilon(v) = \int_\Omega \Big(\frac{1}{2}|\nabla v|^2 + \frac{1}{\varepsilon^2}\chi\big(\operatorname{dist}^2(v, \mathcal{N})\big)\Big).$$

Then the corresponding Ginzburg-Landau approximate system of (1.1) can be written as

$$\begin{cases} u_t + u \cdot \nabla u - \Delta u + \nabla P = -\nabla \cdot \big(\nabla v \odot \nabla v\big), \\ \nabla \cdot u = 0, \\ v_t + u \cdot \nabla v = \Delta v - \dfrac{1}{\varepsilon^2}\chi'\big(\operatorname{dist}^2(v, \mathcal{N})\big)\dfrac{d}{dv}\big(\operatorname{dist}^2(v, \mathcal{N})\big), \end{cases} \quad \text{in } Q_T. \tag{1.4}$$

We would like to remark that due to the difficulty in showing the weak convergence of the Ericksen stress tensors of v^ε to that of a weak limit v, it has been a challenging question to ask if a weak limit (u, v) of $(u^\varepsilon, v^\varepsilon)$ solves the original Ericksen-Leslie system (1.1). See for example [18].

The main purpose of this paper is to establish in dimension two: (i) the convergence of weak solutions of the Ginzburg-Landau approximate system (1.4) to the simplified Ericksen-Leslie system (1.1), and (ii) the weak compactness of weak solutions to the simplified Ericksen-Leslie system (1.1).

To simplify the presentation, we will consider the following initial and boundary condition

$$(u, v)|_{\partial_p Q_T} = (u_0, v_0) \tag{1.5}$$

where $\partial_p Q_T = \big(\Omega \times \{t = 0\}\big) \cup \big(\partial\Omega \times [0, T]\big)$ denotes the parabolic boundary of Q_T. We will assume that $(u_0, v_0) : \Omega \to \mathbb{R}^2 \times \mathbb{R}^L$ satisfies

$$u_0\big|_{\partial\Omega} = 0, \quad v_0(x) \in \mathcal{N} \text{ for a.e. } x \in \Omega. \tag{1.6}$$

We now introduce the following notations

$$\begin{aligned} \mathbf{H} &= \text{Closure of } C_0^\infty\left(\Omega, \mathbb{R}^2\right) \cap \Big\{f \;\Big|\; \nabla \cdot f = 0\Big\} \text{ in } L^2\left(\Omega, \mathbb{R}^2\right), \\ \mathbf{J} &= \text{Closure of } C_0^\infty\left(\Omega, \mathbb{R}^2\right) \cap \Big\{f \;\Big|\; \nabla \cdot f = 0\Big\} \text{ in } H_0^1\left(\Omega, \mathbb{R}^2\right), \\ H^1(\Omega, \mathcal{N}) &= \Big\{f \in H^1(\Omega, \mathbb{R}^L) \;\Big|\; f(x) \in \mathcal{N} \text{ a.e. } x \in \Omega\Big\}. \end{aligned}$$

We will assume that

$$u_0 \in \mathbf{H}, \quad v_0 \in H^1(\Omega, \mathcal{N}). \tag{1.7}$$

Recall the definition of weak solutions of (1.1).

Definition 1.1 Assume (1.6) and (1.7). For $T > 0$, a pair of maps $u \in L^\infty([0, T], \mathbf{H}) \cap L^2([0, T], \mathbf{J})$ and $v \in L^2([0, T], H^1(\Omega, \mathcal{N}))$ is called a weak solution to the initial and boundary problem (1.1) and (1.5), if

$$\begin{aligned} &-\int_{Q_T} \big\langle u, \xi'\varphi\big\rangle + \int_{Q_T} \langle u \cdot \nabla u, \xi\varphi\rangle + \langle \nabla u, \xi\nabla\varphi\rangle \\ &= -\xi(0)\int_\Omega \langle u_0, \varphi\rangle + \int_{Q_T} \langle \nabla v \odot \nabla v, \xi\nabla\varphi\rangle, \\ &-\int_{Q_T} \big\langle v, \xi'\phi\big\rangle + \int_{Q_T} \langle u\nabla v, \xi\phi\rangle + \langle \nabla v, \xi\nabla\phi\rangle \\ &= -\xi(0)\int_\Omega \langle v_0, \phi\rangle + \int_{Q_T} \langle A(v)(\nabla v, \nabla v), \xi\phi\rangle, \end{aligned} \tag{1.8}$$

for any $\xi \in C^\infty([0,T])$ with $\xi(T)=0$, $\varphi \in \mathbf{J}$ and $\phi \in H_0^1(\Omega,\mathbb{R}^L)\cap L^\infty(\Omega,\mathbb{R}^L)$. Moreover, $(u,v)|_{\partial\Omega}=(u_0,v_0)$ holds in the sense of traces. Similarly, the notion of a weak solution to the system (1.4) and (1.5) can be defined for $u \in L^\infty([0,T],\mathbf{H})\cap L^2([0,T],\mathbf{J})$ and $v\in L^2([0,T],H^1(\Omega,\mathbb{R}^L))$.

Our first main theorem concerns the convergence of weak solutions of the system (1.4) to the system (1.1), as $\varepsilon\to 0$. We would like to remark that for any $\varepsilon>0$, the existence of weak solutions to (1.4) has been established by [18, 19] for $\mathcal{N}=\mathbb{S}^2\subset\mathbb{R}^3$ by the Galerkin method, which can be extended without much difficulty to the case that $\mathcal{N}$ is a compact Riemannian manifold.

Theorem 1.2 *Under the assumptions* (1.6) *and* (1.7)*, for $\varepsilon>0$ let $(u^\varepsilon,v^\varepsilon)$ be a sequence of weak solutions to the Ginzburg-Landau approximated system* (1.4) *subject to the initial and boundary condition* (1.5)*. Then there exists a weak solution (u,v) of* (1.1)*, subject to the initial and boundary condition* (1.5) *such that, after passing to a subsequence,*

$$u^\varepsilon \rightharpoonup u \ \text{ in } \ L^2([0,T],H^1(\Omega)), \quad v^\varepsilon \rightharpoonup v \ \text{ in } \ L^2([0,T],H^1(\Omega)).$$

In particular, the initial and boundary problem (1.1) *and* (1.5) *admits at least one weak solution $u\in L^\infty([0,T],\mathbf{H})\cap L^2([0,T],\mathbf{J})$ and $v\in L^2([0,T],H^1(\Omega,\mathcal{N}))$.*

We would like to mention that when $\mathcal{N}=\mathbb{S}^2$, the convergence of weak solutions $(u^\varepsilon,v^\varepsilon)$ of the system (1.4) to a weak solution (u,v) of the system (1.2) has recently been proved in two dimensional torus T^2 by Kortum in an interesting article [12]. In order to deal with the most difficult terms $\nabla v^\varepsilon\odot\nabla v^\varepsilon$ in the limit process, Kortum employed the concentration-cancellation method for the Euler equation developed by DiPerna and Majda [3] (see also [26]). Thanks to the rotational covariance of $\nabla v^\varepsilon\odot\nabla v^\varepsilon$, the test functions can be taken to periodic functions of one spatial variable to verify the convergence $\nabla\cdot(\nabla v^\varepsilon\odot\nabla v^\varepsilon)$ to $\nabla\cdot(\nabla v\odot\nabla v)$ in the dual space of $C^\infty_{0,div}(\Omega,\mathbb{R}^2)$. The paper [12] left it open whether the Ericksen stress tensors $(\nabla v^\varepsilon\odot\nabla v^\varepsilon-\frac12|\nabla v^\varepsilon|^2\mathbb{I}_2)$ weakly converges to $(\nabla v\odot\nabla v-\frac12|\nabla v|^2\mathbb{I}_2)$ as Radon measures.

In this paper, we make some new observations on the Ericksen stress tensor $\nabla v^\varepsilon\odot\nabla v^\varepsilon$, that is flexible enough to treat any smooth domain $\Omega\subset\mathbb{R}^2$. More precisely, by adding $-\frac12|\nabla v^\varepsilon|^2\mathbb{I}_2$ to $\nabla v^\varepsilon\odot\nabla v^\varepsilon$, where $\mathbb{I}_2$ is the 2×2 identity matrix, we have

$$\nabla v^\varepsilon\odot\nabla v^\varepsilon-\frac12|\nabla v^\varepsilon|^2\mathbb{I}_2=\frac12\begin{pmatrix}|\partial_x v^\varepsilon|^2-|\partial_y v^\varepsilon|^2, & 2\langle\partial_x v^\varepsilon,\partial_y v^\varepsilon\rangle\\ 2\langle\partial_x v^\varepsilon,\partial_y v^\varepsilon\rangle, & |\partial_y v^\varepsilon|^2-|\partial_x v^\varepsilon|^2\end{pmatrix}.$$

This is a 2×2 matrix-valued function whose components consist of the Hopf differential of map v^ε, which are $|\partial_x v^\varepsilon|^2-|\partial_y v^\varepsilon|^2$ and $\langle\partial_x v^\varepsilon,\partial_y v^\varepsilon\rangle$. Since v^ϵ is either an approximated harmonic map to $\mathcal{N}$ or an Ginzburg-Landau approximated harmonic map to $\mathcal{N}$, we can develop the compensated compactness property of the Ericksen stress tensors by the Pohozaev argument (see, for example, Schoen [27] or Lin–Wang [20]).

As a byproduct of the proof of Theorem 1.2, we obtain the following compactness for a sequence of weak solutions to the system (1.1).

Theorem 1.3 *Let $(u^k,v^k): Q_T\to\mathbb{R}^2\times\mathcal{N}$ be a sequence of weak solutions to* (1.1)*, along with the initial and boundary condition (u_0^k,v_0^k) satisfying* (1.6)*, such that*

$$\sup_{k\geq 1}\left\{\int_{Q_T}\left(|u^k|^2+|\nabla v^k|^2\right)+\int_{Q_T}\left(|\nabla u^k|^2+|v_t^k+u^k\cdot\nabla v^k|^2\right)\right\}<\infty, \tag{1.9}$$

and if, in addition, we assume

$$(u_0^k, v_0^k) \rightharpoonup (u_0, v_0) \ \ in \ \ L^2(\Omega, \mathbb{R}^2) \times H^1(\Omega, \mathbb{S}^2),$$

then there exists a weak solution (u, v) *of* (1.1) *with the initial and boundary condition* (u_0, v_0) *such that, after passing to subsequences,*

$$u^k \rightharpoonup u \ in \ L^2([0, T], H^1(\Omega)), \qquad v^k \rightharpoonup v \ in \ L^2([0, T], H^1(\Omega)).$$

Since the system (1.1) possesses the geometric structure, i.e.,

$$A(v^k)(\cdot, \cdot) \perp T_{v^k}\mathcal{N}$$

where $T_{v^k}\mathcal{N}$ is the tangent space of $\mathcal{N}$ at v^k, we observe that for a.e. $t \in (0, T)$, $v^k(\cdot, t) : \Omega \to \mathcal{N}$ can be regarded as an approximated harmonic map with L^2-tension field $\tau_k \equiv (v_t^k + u^k \cdot \nabla v^k)(\cdot, t)$. Hence we can show the weak convergence of $(|\partial_x v^k|^2 - |\partial_y v^k|^2)$ to $(|\partial_x v|^2 - |\partial_y v|^2)$ and $\langle \partial_x v^k, \partial_y v^k \rangle$ to $\langle \partial_x v, \partial_y v \rangle$ in L^1 by utilizing the L^p-estimate, $1 < p < 2$, of the Hopf differential of $v^k(\cdot, t)$.

The paper is organized as follows. In section two, we will prove some uniform estimates on v^ε under a smallness condition on the Ginzburg-Landau energy. In section three, we will establish the weak compactness of the Ericksen stress tensors of v^ϵ by the Pohozaev argument and prove Theorem 1.2. In section four, we will prove Theorem 1.3 by establishing the L^p-estimates, $1 < p < 2$, of the Hopf differential of v^k.

2 Uniform estimates of inhomogeneous Ginzburg-Landau equations

In this section, we will establish uniform estimates on v^ϵ under the smallness condition on the Ginzburg-Landau energy. More precisely, we will consider a family of solutions v^ε to the inhomogeneous Ginzburg-Landau equation:

$$\Delta v^\varepsilon - \frac{1}{\varepsilon^2}\chi'\big(\mathrm{dist}^2(v^\varepsilon, \mathcal{N})\big)\frac{d}{dv}\big(\mathrm{dist}^2(v^\varepsilon, \mathcal{N})\big) = \tau_\varepsilon \quad \text{in} \quad \Omega. \tag{2.1}$$

We will assume that there exist $0 < \Lambda_1, \Lambda_2 < \infty$ such that

$$\sup_{0<\varepsilon\le 1} \mathcal{E}_\varepsilon(v^\varepsilon) = \int_\Omega \left(\frac{1}{2}|\nabla v^\varepsilon|^2 + \frac{1}{\varepsilon^2}\chi(\mathrm{dist}^2(v^\varepsilon, \mathcal{N}))\right) \le \Lambda_1 < \infty, \tag{2.2}$$

and

$$\sup_{0<\varepsilon\le 1} \left\|\tau^\varepsilon\right\|_{L^2(\Omega)} \le \Lambda_2 < \infty. \tag{2.3}$$

After passing to a subsequence, we may assume that there exist $v \in H^1(\Omega, \mathcal{N})$ and $\tau \in L^2(\Omega, \mathbb{R}^L)$ such that

$$\tau^\varepsilon \rightharpoonup \tau \ \text{ in } \ L^2(\Omega), \quad v^\varepsilon \rightharpoonup v \ \text{ in } \ H^1(\Omega).$$

Then we have

Lemma 2.1 *There exists* $\delta_0 > 0$ *such that if* $v^\varepsilon \in H^1(\Omega, \mathbb{R}^L)$ *is a family of solutions to* (2.1) *satisfying* (2.2) *and* (2.3), *and if for* $\mathbf{x}_0 \in \Omega$ *and* $0 < r_0 < \mathrm{dist}(x_0, \partial\Omega)$,

$$\sup_{0<\varepsilon\le 1} \int_{B_{r_0}(\mathbf{x}_0)} \big(\frac{1}{2}|\nabla v^\varepsilon|^2 + \frac{1}{\varepsilon^2}\chi\big(\mathrm{dist}^2(v^\varepsilon, \mathcal{N})\big)\big) \le \delta_0^2, \tag{2.4}$$

then there exists an approximated harmonic map $v \in H^1(B_{\frac{r_0}{4}}(\mathbf{x}_0), \mathcal{N})$ *with tension filed* τ*, i.e,*

$$\Delta v + A(v)(\nabla v, \nabla v) = \tau, \tag{2.5}$$

such that as $\epsilon \to 0$*,*

$$v^\varepsilon \to v \quad in \;\; H^1(B_{\frac{r_0}{4}}(\mathbf{x}_0)), \;\; \text{and} \;\; \frac{1}{\varepsilon^2}\chi\Big(\operatorname{dist}^2(v^\varepsilon, \mathcal{N})\Big) \to 0 \;\; \text{in} \;\; L^1(B_{\frac{r_0}{4}}(\mathbf{x}_0)). \tag{2.6}$$

Proof For any fixed $\mathbf{x}_1 \in B_{\frac{r_0}{2}}(\mathbf{x}_0)$ and $0 < \varepsilon \le \frac{r_0}{2}$, define $\widehat{v}^\varepsilon(\mathbf{x}) = v^\varepsilon(\mathbf{x}_1 + \varepsilon\mathbf{x}) : B_1(0) \to \mathbb{R}^L$. Then we have

$$\Delta\widehat{v}^\varepsilon = \chi'(\operatorname{dist}^2(\widehat{v}^\varepsilon, \mathcal{N}))\frac{d}{dv}(\operatorname{dist}^2(\widehat{v}^\varepsilon, \mathcal{N})) + \widehat{\tau}^\varepsilon \text{ in } B_1(0),$$

where $\widehat{\tau}^\varepsilon(\mathbf{x}) = \varepsilon^2\tau^\varepsilon(\mathbf{x}_1 + \varepsilon\mathbf{x})$. Since

$$\begin{aligned}\left\|\Delta\widehat{v}^\varepsilon\right\|_{L^2(B_1(0))} &\le \left\|\chi'(\operatorname{dist}(\widehat{v}^\varepsilon, \mathcal{N}))\frac{d}{dv}(\operatorname{dist}^2(\widehat{v}^\varepsilon, \mathcal{N}))\right\|_{L^2(B_1(0))} + \left\|\widehat{\tau}^\varepsilon\right\|_{L^2(B_1(0))} \\ &\le C\Big(\int_{\Omega\cap\{\operatorname{dist}(v^\varepsilon,\mathcal{N})\le 2\delta_{\mathcal{N}}\}} \operatorname{dist}^2(v^\varepsilon, \mathcal{N})\Big)^{\frac{1}{2}} + \varepsilon\left\|\tau^\varepsilon\right\|_{L^2(\Omega)} \le C + \Lambda_2.\end{aligned}$$

Thus $\widehat{v}^\varepsilon \in H^2(B_{\frac{1}{2}})$ and $\|\widehat{v}^\varepsilon\|_{H^2(B_{\frac{1}{2}})} \le C(1+\Lambda_2)$. Hence by the Sobolev embedding theorem we have that $\widehat{v}^\varepsilon \in C^{\frac{1}{2}}(B_{\frac{1}{2}})$ and

$$[\widehat{v}^\varepsilon]_{C^{\frac{1}{2}}(B_{\frac{1}{2}})} \le C\left\|\widehat{v}^\varepsilon\right\|_{H^2(B_{\frac{1}{2}})} \le C(1+\Lambda_2).$$

By rescaling, we get

$$|v^\varepsilon(\mathbf{x}) - v^\varepsilon(\mathbf{y})| \le C(1+\Lambda_2)\Big(\frac{|\mathbf{x}-\mathbf{y}|}{\varepsilon}\Big)^{\frac{1}{2}}, \qquad \forall \mathbf{x}, \mathbf{y} \in B_\varepsilon(\mathbf{x}_1).$$

We claim that $\operatorname{dist}(v^\varepsilon, \mathcal{N}) \le \delta_{\mathcal{N}}$ on $B_{\frac{r_0}{2}}(\mathbf{x}_0)$. Suppose it were false. Then there exists $\mathbf{x}_1 \in B_{\frac{r_0}{2}}(\mathbf{x}_0)$ such that $\operatorname{dist}(v^\varepsilon(\mathbf{x}_1), \mathcal{N}) > \delta_{\mathcal{N}}$. Then for any $\theta_0 \in (0,1)$ and $\mathbf{x} \in B_{\theta_0\varepsilon}(\mathbf{x}_1)$, it holds

$$|v^\varepsilon(\mathbf{x}) - v^\varepsilon(\mathbf{x}_1)| \le C\Big(\frac{|\mathbf{x}-\mathbf{x}_1|}{\varepsilon}\Big)^{\frac{1}{2}} \le C\theta_0^{\frac{1}{2}} \le \frac{1}{2}\delta_{\mathcal{N}},$$

provided $\theta_0 \le \frac{\delta_{\mathcal{N}}^2}{4C^2}$. It follows that

$$\operatorname{dist}(v^\varepsilon(\mathbf{x}), \mathcal{N}) \ge \frac{1}{2}\delta_{\mathcal{N}}, \;\; \forall \mathbf{x} \in B_{\theta_0\varepsilon}(\mathbf{x}_1),$$

so that

$$\int_{B_{\theta_0\varepsilon}(\mathbf{x}_1)} \frac{1}{\varepsilon^2}\chi(\operatorname{dist}^2(v^\varepsilon, \mathcal{N})) \ge \pi\delta_{\mathcal{N}}^2\theta_0^2.$$

This contradicts the assumption that

$$\int_{B_{\theta_0\varepsilon}(\mathbf{x}_1)} \frac{1}{\varepsilon^2}\chi(\operatorname{dist}^2(v^\varepsilon, \mathcal{N})) \le \int_{B_{r_1}(0)} \Big(\frac{1}{2}|\nabla v^\varepsilon|^2 + \frac{1}{\varepsilon^2}\chi\left(\operatorname{dist}^2(v^\varepsilon, \mathcal{N})\right)\Big) \le \delta_0^2,$$

provided we choose a sufficiently small $\delta_0 > 0$.

From $\mathrm{dist}(v^\varepsilon, \mathcal{N}) \le \delta_{\mathcal{N}}$ in $B_{\frac{r_0}{2}}(\mathbf{x}_0)$, we may decompose v^ε into

$$v^\varepsilon = \Pi_{\mathcal{N}}(v^\varepsilon) + \mathrm{dist}(v^\varepsilon, \mathcal{N})\nu\big(\Pi_{\mathcal{N}}(v^\varepsilon)\big) := \omega_\varepsilon + d_\varepsilon \nu_\varepsilon,$$

where $\omega_\varepsilon = \Pi_{\mathcal{N}}(v^\varepsilon)$, $d_\varepsilon = \mathrm{dist}(v^\varepsilon, \mathcal{N})$, $\nu_\varepsilon = \nu(\Pi_{\mathcal{N}}(v^\varepsilon))$, and

$$\nu(\Pi_{\mathcal{N}}(\mathbf{y})) = \frac{\mathbf{y} - \Pi_{\mathcal{N}}(\mathbf{y})}{|\mathbf{y} - \Pi_{\mathcal{N}}(\mathbf{y})|},\ \mathbf{y} \in \mathcal{N}_{\delta_{\mathcal{N}}} \setminus \mathcal{N},$$

denotes a unit normal vector of $\mathcal{N}$ at the point $\Pi_{\mathcal{N}}(\mathbf{y})$.

Thus the equation of v_ε can be rewritten as

$$\Delta\omega_\varepsilon + \Delta d_\varepsilon \nu_\varepsilon + 2\nabla d_\varepsilon \nabla \nu_\varepsilon + d_\varepsilon \Delta \nu_\varepsilon - \frac{1}{\varepsilon^2}\chi'(d_\varepsilon^2)\nabla_{\nu_\varepsilon} d_\varepsilon^2 = \tau_\varepsilon. \tag{2.7}$$

Multiplying (2.7) by ν_ε, we get

$$\Delta d_\varepsilon = \langle \nabla\omega_\varepsilon, \nabla\nu_\varepsilon\rangle + d_\varepsilon|\nabla\nu_\varepsilon|^2 + \frac{2}{\varepsilon^2}\chi'(d_\varepsilon^2)d_\varepsilon + \tau_\varepsilon^\perp, \tag{2.8}$$

where $\tau_\varepsilon^\perp = \langle \tau_\varepsilon, \nu_\varepsilon\rangle$. Plugging Δd_ε into (2.7), we obtain

$$\Delta\omega_\varepsilon + \langle \nabla\omega_\varepsilon, \nabla\nu_\varepsilon\rangle\nu_\varepsilon + d_\varepsilon\big(\Delta\nu_\varepsilon + |\nabla\nu_\varepsilon|^2\nu_\varepsilon\big) + 2\langle\nabla\nu_\varepsilon, \nabla d_\varepsilon\rangle = \tau_\varepsilon'', \tag{2.9}$$

where $\tau_\varepsilon'' = \tau_\varepsilon - \tau_\varepsilon^\perp\nu_\varepsilon$. Here we have used the identities:

$$\langle \nabla_{\nu_\varepsilon} d_\varepsilon^2, \nu_\varepsilon\rangle = 2d_\varepsilon, \quad \langle \nabla_{\nu_\varepsilon} d_\varepsilon^2, \nu_\varepsilon\rangle\nu_\varepsilon = \nabla_{\nu_\varepsilon} d_\varepsilon^2.$$

Let $\eta \in C_0^\infty(B_{\frac{r_0}{2}}(\mathbf{x}_0), \mathbb{R})$ be a standard cutoff function of $B_{\frac{3r_0}{8}}(\mathbf{x}_0)$. Since $\mathrm{dist}(v^\varepsilon, \mathcal{N}) \le \delta_{\mathcal{N}}$ on $B_{r_0}(\mathbf{x}_0)$, it follows that $d_\varepsilon = \mathrm{dist}(v^\varepsilon, \mathcal{N}) \le \delta_{\mathcal{N}}$ on $B_{r_0}(\mathbf{x}_0)$, i.e. $\|d_\varepsilon\|_{L^\infty(B_{r_0}(\mathbf{x}_0))} \le \delta_{\mathcal{N}}$, so that $\chi'(d_\varepsilon^2) = 1$ and hence

$$\begin{aligned}\left(-\Delta + \frac{2}{\varepsilon^2}\right)(d_\varepsilon\eta^2) =& - d_\varepsilon\Delta(\eta^2) - 2\nabla d_\varepsilon\nabla(\eta^2) + \langle\nabla\omega_\varepsilon, \nabla(\nu_\varepsilon\eta^2)\rangle - \langle\nabla\omega_\varepsilon, \nu_\varepsilon\nabla(\eta^2)\rangle \\ &+ d_\varepsilon\left(|\nabla(\nu_\varepsilon\eta^2)|^2 - |\nu_\varepsilon\nabla(\eta^2)|^2\right) + \tau_\varepsilon^\perp\eta^2.\end{aligned} \tag{2.10}$$

For sufficiently small $\varepsilon > 0$, by applying the $W^{2,\frac{4}{3}}$-estimate for $(-\Delta + \frac{2}{\varepsilon^2})$ uniformly in ε (see [13]), we obtain that

$$\begin{aligned}&\|\nabla^2(d_\varepsilon\eta^2)\|_{L^{\frac{4}{3}}} \\ &\lesssim \|d_\varepsilon\Delta(\eta^2)\|_{L^{\frac{4}{3}}} + \|\nabla d_\varepsilon\nabla(\eta^2)\|_{L^{\frac{4}{3}}} + \|\nabla\omega_\varepsilon\|_{L^2}\|\nabla(\nu_\varepsilon\eta^2)\|_{L^4} \\ &\quad + \|\nabla\omega_\varepsilon\|_{L^2} + \|d_\varepsilon\|_{L^\infty}\|\nabla\nu_\varepsilon\|_{L^2}\|\nabla(\nu_\varepsilon\eta^2)\|_{L^4} + \|\tau_\varepsilon^\perp\|_{L^{\frac{4}{3}}} \\ &\lesssim \|d_\varepsilon\|_{L^\infty} + \|\nabla d_\varepsilon\|_{L^2} + \|\nabla\omega_\varepsilon\|_{L^2}(\|\nabla(\nu_\varepsilon\eta^2)\|_{L^4} + 1) \\ &\quad + \|d_\varepsilon\|_{L^\infty}\|\nabla\nu_\varepsilon\|_{L^2}\|\nabla(\nu_\varepsilon\eta^2)\|_{L^4} + \|\tau_\varepsilon\|_{L^2},\end{aligned} \tag{2.11}$$

where $A \lesssim B$ stands for $A \le CB$ for some universal positive constant C.

For ω_ε, by a similar calculation we obtain

$$\begin{aligned}\Delta(\omega_\varepsilon\eta^2) = & - \langle\nabla\omega_\varepsilon, \nabla(\nu_\varepsilon\eta^2)\rangle\nu_\varepsilon + \langle\nabla\omega_\varepsilon, \nu_\varepsilon\nabla(\eta^2)\rangle\nu_\varepsilon \\ & - d_\varepsilon\big[\Delta(\nu_\varepsilon\eta^2) - \nu_\varepsilon\Delta(\eta^2) - 2\nabla\nu_\varepsilon\nabla(\eta^2)\big] + d_\varepsilon\big[|\nabla\nu_\varepsilon\eta^2|^2 - |\nu_\varepsilon\nabla(\eta^2)|^2\big]\nu_\varepsilon \\ & - 2\big[\langle\nabla(\nu_\varepsilon\eta^2), \nabla d_\varepsilon\rangle - \langle\nabla(\eta^2), \nabla d_\varepsilon\rangle\nu_\varepsilon\big] + \tau_\varepsilon''\eta^2 + \omega_\varepsilon\Delta(\eta^2) + 2\nabla\omega_\varepsilon\nabla(\eta^2).\end{aligned} \tag{2.12}$$

Applying the $W^{2,\frac{4}{3}}$-estimate, we obtain

$$\begin{aligned} & \|\nabla^2(\omega_\varepsilon\eta^2)\|_{L^{\frac{4}{3}}} \\ & \lesssim \|\nabla\omega_\varepsilon\|_{L^2}\|\nabla(\nu_\varepsilon\eta^2)\|_{L^4} + \|\nabla\omega_\varepsilon\|_{L^{\frac{4}{3}}} + \|d_\varepsilon\|_{L^\infty}\|\Delta(\nu_\varepsilon\eta^2)\|_{L^{\frac{4}{3}}} \\ & \quad + \|d_\varepsilon\|_{L^\infty}\left(1 + \|\nabla\nu_\varepsilon\|_{L^{\frac{4}{3}}}\right) + \|d_\varepsilon\|_{L^\infty}\|\nabla(\nu_\varepsilon\eta^2)\|_{L^2}\|\nabla(\nu_\varepsilon\eta^2)\|_{L^4} \\ & \quad + \|\nabla d_\varepsilon\|_{L^2}\|\nabla(\nu_\varepsilon\eta^2)\|_{L^4} + \|\nabla d_\varepsilon\|_{L^{\frac{4}{3}}} + \|\tau_\varepsilon\|_{L^2}.\end{aligned} \tag{2.13}$$

Therefore, we conclude that

$$\begin{aligned} & \|\nabla^2(d_\varepsilon\eta^2)\|_{L^{\frac{4}{3}}} + \|\nabla^2(\omega_\varepsilon\eta^2)\|_{L^{\frac{4}{3}}} \\ & \lesssim \|\nabla v^\varepsilon\|_{L^2}\|\nabla(\nu_\varepsilon\eta^2)\|_{L^4} + \|d_\varepsilon\|_{L^\infty}\|\nabla^2(\nu_\varepsilon\eta^2)\|_{L^{\frac{4}{3}}} + \|\nabla v^\varepsilon\|_{L^2} + \|\tau_\varepsilon\|_{L^2}.\end{aligned} \tag{2.14}$$

Since

$$v^\varepsilon\eta^2 = \omega_\varepsilon\eta^2 + d_\varepsilon\nu_\varepsilon\eta^2$$

we have

$$\begin{aligned} & \|\nabla^2(v^\varepsilon\eta^2)\|_{L^{\frac{4}{3}}} \\ & \lesssim \|\nabla^2(d_\varepsilon\nu_\varepsilon\eta^2)\|_{L^{\frac{4}{3}}} + \|\nabla^2(\omega_\varepsilon\eta^2)\|_{L^{\frac{4}{3}}} \\ & \lesssim \|d_\varepsilon\|_{L^\infty}\|\nabla^2(\nu_\varepsilon\eta^2)\|_{L^{\frac{4}{3}}} + \|\nabla d_\varepsilon\|_{L^2}\|\nabla(\nu_\varepsilon\eta^2)\|_{L^4} + \|\nabla^2 d_\varepsilon\eta^2\|_{L^{\frac{4}{3}}} + \|\nabla^2(\omega_\varepsilon\eta^2)\|_{L^{\frac{4}{3}}} \\ & \lesssim \|d_\varepsilon\|_{L^\infty}\|\nabla^2(\nu_\varepsilon\eta^2)\|_{L^{\frac{4}{3}}} + \|\nabla d_\varepsilon\|_{L^2}\|\nabla(\nu_\varepsilon\eta^2)\|_{L^4} + \|\nabla^2(d_\varepsilon\eta^2)\|_{L^{\frac{4}{3}}} \\ & \quad + \|\nabla d_\varepsilon\|_{L^2} + \|\nabla^2(w_\varepsilon\eta^2)\|_{L^{\frac{4}{3}}} + 1.\end{aligned} \tag{2.15}$$

Therefore, we have

$$\begin{aligned} & \|\nabla^2(v^\varepsilon\eta^2)\|_{L^{\frac{4}{3}}} \\ & \lesssim \|d_\varepsilon\|_{L^\infty}\|\nabla^2(\nu_\varepsilon\eta^2)\|_{L^{\frac{4}{3}}} + \|\nabla v^\varepsilon\|_{L^2}\left[1 + \|\nabla(v^\varepsilon\eta^2)\|_{L^4} + \|\nabla(\nu_\varepsilon\eta^2)\|_{L^4}\right] + \|\tau_\varepsilon\|_{L^2} + 1 \\ & \lesssim \|d_\varepsilon\|_{L^\infty}\|\nabla^2(\nu_\varepsilon\eta^2)\|_{L^{\frac{4}{3}}} + \|\nabla v^\varepsilon\|_{L^2}\left[1 + \|\nabla(v^\varepsilon\eta^2)\|_{L^4}\right] + \|\tau_\varepsilon\|_{L^2} + 1.\end{aligned} \tag{2.16}$$

Since $\nu_\varepsilon = \nu_\varepsilon(v^\varepsilon)$, we can directly calculate and show that

$$\|\nabla^2(\nu_\varepsilon\eta^2)\|_{L^{\frac{4}{3}}} \lesssim \|\nabla^2(v^\varepsilon\eta^2)\|_{L^{\frac{4}{3}}} + \|\nabla v^\varepsilon\|_{L^2}\left[1 + \|\nabla(v^\varepsilon\eta^2)\|_{L^4}\right] + 1. \tag{2.17}$$

Therefore, we can conclude that

$$(1 - C\|d_\varepsilon\|_{L^\infty}) \|\nabla^2(v^\varepsilon \eta^2)\|_{L^{\frac{4}{3}}} \lesssim \|\nabla v^\varepsilon\|_{L^2} \left[1 + \|\nabla(v^\varepsilon \eta^2)\|_{L^4}\right] + \|\tau_\varepsilon\|_{L^2} + 1. \quad (2.18)$$

Since $\|d_\varepsilon\|_{L^\infty} \le \delta_{\mathcal{N}}$, we have that

$$1 - C\|d_\varepsilon\|_{L^\infty} \ge 1 - C\delta_{\mathcal{N}} \ge \frac{1}{2},$$

provided $\delta_{\mathcal{N}}$ is chosen sufficiently small. From this and Sobolev's embedding, (2.18) implies

$$\|\nabla(v^\varepsilon \eta^2)\|_{L^4} \lesssim \|\nabla v^\varepsilon\|_{L^2} \left[1 + \|\nabla(v^\varepsilon \eta^2)\|_{L^4}\right] + \|\tau_\varepsilon\|_{L^2} + 1. \quad (2.19)$$

Taking δ_0 small enough in the assumption (2.4), we conclude that

$$\|\nabla(v^\varepsilon \eta^2)\|_{L^4} \lesssim \|\nabla v^\varepsilon\|_{L^2} + \|\tau_\varepsilon\|_{L^2} + 1 \le C(\delta_0, \Lambda_2). \quad (2.20)$$

Substituting this into (2.19), we obtain that

$$\|\nabla^2(v_\varepsilon \eta^2)\|_{L^{\frac{4}{3}}} \le C(\delta_0, \Lambda_2). \quad (2.21)$$

Hence $v^\varepsilon \to v$ in $H^1(B_{\frac{r_0}{3}}(\mathbf{x}_0))$.

By Fubini's theorem, there exists $r_1 \in [\frac{r_0}{4}, \frac{r_0}{3}]$ such that

$$\int_{\partial B_{r_1}(\mathbf{x}_0)} |\nabla d_\varepsilon|^2 \le C \int_{B_{\frac{r_0}{3}}(\mathbf{x}_0)} |\nabla d_\varepsilon|^2 \le C, \quad \int_{\partial B_{r_1}(\mathbf{x}_0)} |d_\varepsilon|^2 \le C \int_{B_{\frac{r_0}{3}}(\mathbf{x}_0)} |d_\varepsilon|^2 \le C\varepsilon^2. \quad (2.22)$$

Multiplying the equation of d_ε by d_ε and integrating by parts over B_{r_1}, we obtain

$$\int_{B_{r_1}(\mathbf{x}_0)} \big(|\nabla d_\varepsilon|^2 + \frac{2}{\varepsilon^2}\chi'(d_\varepsilon) d_\varepsilon^2 + |\nabla v_\varepsilon|^2 d_\varepsilon^2 + \nabla\omega_\varepsilon \nabla v_\varepsilon \cdot d_\varepsilon\big) - \int_{\partial B_{r_1}(\mathbf{x}_0)} \frac{\partial d_\varepsilon}{\partial \nu} d_\varepsilon = \int_{B_{r_1}(\mathbf{x}_0)} \tau_\varepsilon^\perp d_\varepsilon. \quad (2.23)$$

Then we have

$$\begin{aligned} &\int_{B_{r_1}(\mathbf{x}_0)} \big(|\nabla d_\varepsilon|^2 + \frac{2}{\varepsilon^2} d_\varepsilon^2\big) \\ &\le C\Big(\int_{\partial B_{r_1}(\mathbf{x}_0)} |\nabla d_\varepsilon|^2\Big)^{\frac{1}{2}} \Big(\int_{\partial B_{r_1}(\mathbf{x}_0)} |d_\varepsilon|^2\Big)^{\frac{1}{2}} + C\Big(\int_{B_{r_1}(\mathbf{x}_0)} |\nabla\omega_\varepsilon|^4\Big)^{\frac{1}{2}} \Big(\int_{B_{r_1}(\mathbf{x}_0)} |d_\varepsilon|^4\Big)^{\frac{1}{2}} \\ &\quad + C\Big(\int_{B_{r_1}(\mathbf{x}_0)} |\tau_\varepsilon|^2\Big)^{\frac{1}{2}} \Big(\int_{B_{r_1}(\mathbf{x}_0)} |d_\varepsilon|^2\Big)^{\frac{1}{2}} \le C\varepsilon. \end{aligned} \quad (2.24)$$

Therefore we have that

$$\frac{d_\varepsilon^2}{\varepsilon^2} \to 0 \text{ in } L^1(B_{r_1}(\mathbf{x}_0)). \quad (2.25)$$

This completes the proof.

Now we define the concentration set by

$$\Sigma := \bigcap_{r>0} \Big\{x \in \Omega : \liminf_{k\to\infty} \int_{B_r(\mathbf{x})} \big(\frac{1}{2}|\nabla v^\varepsilon|^2 + \frac{1}{\varepsilon^2}\chi\big(\mathrm{dist}^2(v^\varepsilon, \mathcal{N})\big)\big) > \delta_0^2\Big\}, \quad (2.26)$$

where $\delta_0 > 0$ is given in Lemma 2.1. We have

Lemma 2.2 *Σ is a finite set, and*

$$v^{\varepsilon} \to v \quad in \quad H^1_{\mathrm{loc}}(\Omega \setminus \Sigma). \tag{2.27}$$

The finiteness of Σ follows from the condition (2.2) and a simple covering argument, see also [24] and [12].

3 Convergence of Ginzburg-Landau approximate solutions

The section is devoted to the proof of Theorem 1.2, which is based on a priori estimates from section 2 and the compensated compactness property of the Ericksen stress tensors.

First, it follows from the global energy inequality of (1.4) that for almost every $t \in (0, T)$,

$$\int_{\Omega\times\{t\}} \left(|u^{\varepsilon}|^2 + |\nabla v^{\varepsilon}|^2 + \frac{1}{\varepsilon^2}\chi\left(\mathrm{dist}^2(v^{\varepsilon}, \mathcal{N})\right)\right) + 2\int_{Q_t} \left(|\nabla u^{\varepsilon}|^2 + \left|v^{\varepsilon}_t + u^{\varepsilon}\cdot\nabla v^{\varepsilon}\right|^2\right) \le E_0. \tag{3.1}$$

Here $E_0 = \int_{\Omega}(|u_0|^2 + |\nabla v_0|^2)$. This, combined with the system (1.4), implies that there exists $1 < p < 2$ such that

$$\sup_{\varepsilon>0}\left[\left\|u^{\varepsilon}_t\right\|_{L^2_t W^{-2,p}_{\mathrm{div}}} + \left\|v^{\varepsilon}_t\right\|_{L^{4/3}_t L^{4/3}_x}\right] < \infty, \tag{3.2}$$

where $W^{-2,p}_{x,\mathrm{div}}$ stands for the dual of $W^{2,p'}_{\mathrm{div}}(\Omega) = \{g \in W^{2,p'}_0(\Omega, \mathbb{R}^2) : \nabla\cdot g = 0\}$ with $p' = p/(p-1)$. To see (3.2), first observe from (3.1) that for $1 < p < 2$,

$$-u^{\varepsilon}\cdot\nabla u^{\varepsilon} + \Delta u^{\varepsilon} - \nabla\cdot(\nabla v^{\varepsilon}\odot\nabla v^{\varepsilon}) \in L^2_t W^{-2,p}_x,$$

and

$$\begin{aligned}
&\left\| -u^{\varepsilon}\cdot\nabla u^{\varepsilon} + \Delta u^{\varepsilon} - \nabla\cdot(\nabla v^{\varepsilon}\odot\nabla v^{\varepsilon})\right\|_{L^2_t W^{-2,p}_x}\\
&\le C\left(\|u^{\varepsilon}\otimes u^{\varepsilon}\|_{L^2_t L^1_x} + \|u^{\varepsilon}\|_{L^2_t L^2_x} + \|\nabla v^{\varepsilon}\odot\nabla v^{\varepsilon}\|_{L^2_t L^1_x}\right)\\
&\le C\left[(1+\|u^{\varepsilon}\|_{L^{\infty}_t L^2_x})\|u^{\varepsilon}\|_{L^{\infty}_t L^2_x} + \|\nabla v^{\varepsilon}\|^2_{L^{\infty}_t L^2_x}\right] \le C.
\end{aligned}$$

Then we can estimate

$$\begin{aligned}
\left\|u^{\varepsilon}_t(\cdot,t)\right\|_{W^{-2,p}_{x,\mathrm{div}}} &= \sup\left\{\langle u^{\varepsilon}_t(\cdot,t), g\rangle : \|g\|_{W^{2,p'}_{\mathrm{div}}} \le 1\right\}\\
&= \sup\left\{\langle -u^{\varepsilon}\cdot\nabla u^{\varepsilon} + \Delta u^{\varepsilon} - \nabla\cdot(\nabla v^{\varepsilon}\odot\nabla v^{\varepsilon}), g\rangle : g \in W^{2,p'}_{\mathrm{div}}(\Omega),\ \|g\|_{W^{2,p'}_0} \le 1\right\}\\
&\le \sup\left\{\langle -u^{\varepsilon}\cdot\nabla u^{\varepsilon} + \Delta u^{\varepsilon} - \nabla\cdot(\nabla v^{\varepsilon}\odot\nabla v^{\varepsilon}), g\rangle : g \in W^{2,p'}_0(\Omega),\ \|g\|_{W^{2,p'}_0} \le 1\right\}\\
&= \left\| -u^{\varepsilon}\cdot\nabla u^{\varepsilon} + \Delta u^{\varepsilon} - \nabla\cdot(\nabla v^{\varepsilon}\odot\nabla v^{\varepsilon})\right\|_{W^{-2,p}_x} \le C.
\end{aligned}$$

To estimate v^{ε}_t, observe that $v^{\varepsilon}_t = (v^{\varepsilon}_t + u^{\varepsilon}\cdot\nabla v^{\varepsilon}) - u^{\varepsilon}\cdot\nabla v^{\varepsilon} \in L^2_t L^2_x + L^4_t L^4_x \times L^2_t L^2_x \subset L^{\frac{4}{3}}_t L^{\frac{4}{3}}_x$, and

$$\left\|v^{\varepsilon}_t\right\|_{L^{\frac{4}{3}}_t L^{\frac{4}{3}}_x} \le \left\|v^{\varepsilon}_t + u^{\varepsilon}\cdot\nabla v^{\varepsilon}\right\|_{L^2_t L^2_x} + \left\|u^{\varepsilon}\right\|_{L^4_t L^4_x}\left\|\nabla v^{\varepsilon}\right\|_{L^2_t L^2_x} \le C,$$

where we have applied (3.1) and the fact that $L^{\infty}_t L^2_x \cap L^2_t H^1_x \subset L^4_t L^4_x$.

Hence by Aubin-Lions' Lemma there exist $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_T, \mathbb{R}^2)$ and $v \in L_t^\infty H_x^1(Q_T, \mathcal{N})$ such that after taking a subsequence,

$$(u^\varepsilon, v^\varepsilon) \to (u, v) \text{ in } L^2(Q_T), \quad (\nabla u^\varepsilon, \nabla v^\varepsilon) \rightharpoonup (\nabla u, \nabla v) \text{ in } L^2(Q_T).$$

This, combined this with (3.1), also implies that

$$v_t^\varepsilon + u^\varepsilon \cdot \nabla v^\varepsilon \rightharpoonup v_t + u \cdot \nabla v \text{ in } L^2(Q_T). \tag{3.3}$$

Furthermore, it follows from (3.2) that $(u^\varepsilon, v^\varepsilon)$ is bounded in $C_w([0, T], W^{-2,p}(\Omega) \times L^{\frac{4}{3}}(\Omega))$. This, together with the boundedness of $(u^\varepsilon, v^\varepsilon) \in L_t^\infty L_x^2(Q_T) \times L_t^\infty H_x^1(\Omega_T)$, enables us to apply [30] Lemma 6 to conclude that $(u^\varepsilon, v^\varepsilon)$ is bounded in $C_w([0, T], L^2(\Omega) \times H^1(\Omega))$. Thus, after taking a possible subsequence, we can assume that

$$\big(u^\varepsilon(t), v^\varepsilon(t)\big) \rightharpoonup (u(t), v(t)) \quad \text{in} \quad L^2(\Omega) \times H^1(\Omega), \tag{3.4}$$

for all $t \in [0, T]$.

By the lower semi-continuity, we have that for all $t \in (0, T)$,

$$\begin{aligned} &\int_{\Omega\times\{t\}} (|u|^2 + |\nabla v|^2) + \int_{Q_t} (|\nabla u|^2 + |v_t + u \cdot \nabla v|^2) \\ &\le \liminf_{\varepsilon\to 0} \Big(\int_{\Omega\times\{t\}} (|u^\varepsilon|^2 + |\nabla v^\varepsilon|^2) + \int_{Q_t} (|\nabla u^\varepsilon|^2 + |v_t^\varepsilon + u^\varepsilon \cdot \nabla v^\varepsilon|^2)\Big) \\ &\le E_0. \end{aligned} \tag{3.5}$$

By Fatou's Lemma, we also have that for all $t \in (0, T)$,

$$\int_0^t \liminf_{\varepsilon\to 0} \int_\Omega (|\nabla u^\varepsilon|^2 + |v_t^\varepsilon + u^\varepsilon \cdot \nabla v^\varepsilon|^2) \le \liminf_{\varepsilon\to 0} \int_0^t \int_\Omega (|\nabla u^\varepsilon|^2 + |v_t^\varepsilon + u^\varepsilon \cdot \nabla v^\varepsilon|^2) \le E_0. \tag{3.6}$$

Hence there exists $A \subset [0, T]$, with Lebesgue measure $|A| = T$, such that for any $t \in A$,

$$\liminf_{\varepsilon\to 0^+} \int_\Omega \left(|\nabla u^\varepsilon|^2 + |v_t^\varepsilon + u^\varepsilon \cdot \nabla v^\varepsilon|^2\right)(t) < \infty. \tag{3.7}$$

Now we define the concentration set at $t \in A$ by

$$\Sigma_t := \bigcap_{r>0} \Big\{ \mathbf{x} \in \Omega : \liminf_{\varepsilon\to 0} \int_{B_r(\mathbf{x})\times\{t\}} \Big(\frac{1}{2}|\nabla v^\varepsilon|^2 + \frac{1}{\varepsilon^2}\chi\big(\mathrm{dist}^2(v^\varepsilon, \mathcal{N})\big)\Big) > \delta_0^2 \Big\}, \tag{3.8}$$

where δ_0 is given by Lemma 2.1. By Lemma 2.2, it holds that $\#(\Sigma_t) \le C(E_0)$, and

$$v^\varepsilon(t) \to v(t) \text{ in } H^1_{\mathrm{loc}}(\Omega \setminus \Sigma(t)).$$

Now we would like to show that v is a weak solution of $(1.1)_3$ by utilizing the geometric structure as in [1] (see also [21]). First notice that there exists a unit vector $\nu_{\mathcal{N}}^\varepsilon \perp T_{\Pi_{\mathcal{N}}(v^\varepsilon)}\mathcal{N}$ such that

$$\frac{d}{dv}\chi\big(\mathrm{dist}^2(v^\varepsilon, \mathcal{N})\big) = 2\chi'(\mathrm{dist}^2(v^\varepsilon, \mathcal{N}))\,\mathrm{dist}(v^\varepsilon, \mathcal{N})\nu_{\mathcal{N}}^\varepsilon.$$

Next, since $\Pi_{\mathcal{N}} : \mathcal{N}_{\delta_{\mathcal{N}}} \to \mathcal{N}$ is the nearest point projection map, it follows from the differential geometry (c.f. [29]) that for any $y \in \mathcal{N}$, the differential map $D\Pi_{\mathcal{N}}(y) \equiv D_y\Pi_{\mathcal{N}}(y) : \mathbb{R}^L \to T_y\mathcal{N}$ is an orthogonal projection map, and $D^2\Pi_{\mathcal{N}}(y)$ is the second fundamental form of $\mathcal{N}$ at $y \in \mathcal{N}$, i.e.,

$$D^2\Pi_{\mathcal{N}}(y)(w_1, w_2) = -A_{\mathcal{N}}(y)(w_1, w_2), \ \forall w_1, w_2 \in T_y\mathcal{N}.$$

In particular,

$$\langle D\Pi_{\mathcal{N}}(\Pi_{\mathcal{N}}(v^{\varepsilon}))(v), \frac{d}{dv}\chi\big(\operatorname{dist}^2(v^{\varepsilon}, \mathcal{N})\big)\rangle = 0$$

holds everywhere in Ω for any vector $v \in \mathbb{R}^L$. Thus for any $t \in A$ and $\phi \in C_0^{\infty}(\Omega \backslash \Sigma_t, \mathbb{R}^L)$, it holds

$$\begin{aligned}
&\int_{\Omega\times\{t\}} \langle v_t^{\varepsilon} + u^{\varepsilon}\cdot\nabla v^{\varepsilon} - \Delta v^{\varepsilon}, D\Pi_{\mathcal{N}}(\Pi_{\mathcal{N}}(v^{\varepsilon}))\phi\rangle \\
&\quad = -\frac{1}{\varepsilon^2}\int_{\Omega\times\{t\}} \langle \frac{d}{dv}\chi\big(\operatorname{dist}^2(v^{\varepsilon}, \mathcal{N})\big), D\Pi_{\mathcal{N}}(\Pi_{\mathcal{N}}(v^{\varepsilon}))\phi\rangle = 0.
\end{aligned} \tag{3.9}$$

From Lemma 2.2, we have that $\nabla v^{\varepsilon} \to \nabla v$ in $L^2_{\mathrm{loc}}(\Omega \setminus \Sigma_t)$. This implies that

$$\begin{aligned}
&\int_{\Omega\times\{t\}} \langle -\Delta v^{\varepsilon}, D\Pi_{\mathcal{N}}(\Pi_{\mathcal{N}}(v^{\varepsilon}))\phi\rangle = \int_{\Omega\times\{t\}} \langle \nabla v^{\varepsilon}, \nabla\big(D\Pi_{\mathcal{N}}(\Pi_{\mathcal{N}}(v^{\varepsilon}))\phi\big)\rangle \\
&\quad \to \int_{\Omega\times\{t\}} \langle \nabla v, \nabla\big(D\Pi_{\mathcal{N}}(\Pi_{\mathcal{N}}(v))\phi\big)\rangle = \int_{\Omega\times\{t\}} \langle \nabla v, \nabla\big(D\Pi_{\mathcal{N}}(v)\phi\big)\rangle \\
&\quad = \int_{\Omega\times\{t\}} (\langle \nabla v, \nabla\phi\rangle - A_{\mathcal{N}}(v)(\nabla v, \nabla v)\phi),
\end{aligned} \tag{3.10}$$

as $\varepsilon \to 0$. Here we have used the fact that

$$D\Pi_{\mathcal{N}}(\Pi_{\mathcal{N}}(v^{\varepsilon}))\phi \to D\Pi_{\mathcal{N}}(\Pi_{\mathcal{N}}(v))\phi = D\Pi_{\mathcal{N}}(v)\phi,$$

in $H^1_{\mathrm{loc}}(\Omega \setminus \Sigma_t)$, as $\varepsilon \to 0$, where $\Pi_{\mathcal{N}}(v) = v$ follows from $v(\Omega) \subset \mathcal{N}$. And in the last step we have used $\nabla\big(D\Pi_{\mathcal{N}}(v)\phi\big) = D^2\Pi_{\mathcal{N}}(v)\nabla v\phi + D\Pi_{\mathcal{N}}(v)\nabla\phi$, which implies

$$\begin{aligned}
\int_{\Omega\times\{t\}} \langle \nabla v, \nabla\big(D\Pi_{\mathcal{N}}(v)\phi\big)\rangle &= \int_{\Omega\times\{t\}} \langle \nabla v, D^2\Pi_{\mathcal{N}}(v)\nabla v\phi + D\Pi_{\mathcal{N}}(v)\nabla\phi\rangle \\
&= \int_{\Omega\times\{t\}} \langle \nabla v, \nabla\phi\rangle - A_{\mathcal{N}}(v)(\nabla v, \nabla v)\phi,
\end{aligned}$$

where we have also used $\partial_i v \in T_v\mathcal{N}$ implying $\langle \nabla v, D\Pi_{\mathcal{N}}(v)\nabla\phi\rangle = \langle \nabla v, \nabla\phi\rangle$.

From (3.7), we may assume that there exists $\tau(t) \in L^2(\Omega, \mathbb{R}^L)$ such that

$$(v_t^{\varepsilon} + u^{\varepsilon}\cdot\nabla v^{\varepsilon})(t) \rightharpoonup \tau(t) \tag{3.11}$$

in $L^2(\Omega)$ as $\varepsilon \to 0$. Substituting this convergence and (3.10) into (3.9), we obtain

$$\int_{\Omega\times\{t\}} \langle \tau(\cdot, t), D\Pi_{\mathcal{N}}(v)\phi\rangle = \int_{\Omega\times\{t\}} \langle \nabla v, \nabla\phi\rangle - A_{\mathcal{N}}(v)(\nabla v, \nabla v)\phi \tag{3.12}$$

for any $t \in A$ and $\phi \in C_0^{\infty}(\Omega \setminus \Sigma_t, \mathbb{R}^L)$. This implies that $v(t)$ is a weak solution of the equation of approximated harmonic maps, with tension field $\tau(t)$, in $\Omega \setminus \Sigma_t$:

$$\Delta v(t) + A_{\mathcal{N}}(v(t))(\nabla v(t), \nabla v(t)) = \tau(t).$$

Since $\tau(t) \in L^2(\Omega, \mathbb{R}^L)$, it follows from the $W^{2,2}$-regularity of approximated harmonic maps in dimension two (see [31] and [28]) that $v(t) \in W^{2,2}_{\mathrm{loc}}(\Omega \setminus \Sigma_t, \mathcal{N})$. Hence $\Delta v(t) + A_{\mathcal{N}}(v(t))(\nabla v(t), \nabla v(t)) \in T_{v(t)}\mathcal{N}$ holds a.e. in $\Omega \setminus \Sigma_t$, which implies that $\tau(t) \in T_{v(t)}\mathcal{N}$ holds a.e. in $\Omega \setminus \Sigma_t$.

Since $\Sigma_t \subset \mathbb{R}^2$ is a finite set, the 2-capacity of Σ_t is zero (see [5]). Hence there exist a sequence $\{\eta_k\}_{k=1}^{\infty} \subset C_0^{\infty}(\Omega)$ such that for all $k \in \mathbb{N}$, $0 \le \eta_k \le 1$, $\Sigma_t \subset \mathrm{int}\{\eta_k = 1\}$, and

$$\lim_{k\to\infty} \int_{\Omega} (|\eta_k|^2 + |\nabla \eta_k|^2)\, dx = 0.$$

Now for any test function $\psi \in C_0^{\infty}(\Omega, \mathbb{R}^L)$, set $\phi_k = \psi(1-\eta_k)$. Then $\phi_k \in C_0^{\infty}(\Omega \setminus \Sigma_t, \mathbb{R}^L)$. Applying (3.12) with ϕ replaced by ϕ_k, we obtain

$$\begin{aligned}\int_{\Omega\times\{t\}} \langle \tau(\cdot,t), D\Pi_{\mathcal{N}}(v)\psi(1-\eta_k)\rangle = & \int_{\Omega\times\{t\}} \langle \nabla v, (1-\eta_k)\nabla\psi - \psi\nabla\eta_k\rangle \\ & -A_{\mathcal{N}}(v)(\nabla v, \nabla v)\psi(1-\eta_k).\end{aligned}$$

It is readily seen that after sending $k \to \infty$, it holds

$$\begin{aligned}&\int_{\Omega\times\{t\}} \langle \tau(\cdot,t), D\Pi_{\mathcal{N}}(v)\psi(1-\eta_k)\rangle \to \int_{\Omega\times\{t\}} \langle \tau(\cdot,t), D\Pi_{\mathcal{N}}(v)\psi\rangle,\\ &\int_{\Omega\times\{t\}} \langle \nabla v, (1-\eta_k)\nabla\psi\rangle \to \int_{\Omega\times\{t\}} \langle \nabla v, \nabla\psi\rangle,\\ &\int_{\Omega\times\{t\}} A_{\mathcal{N}}(v)(\nabla v, \nabla v)\psi(1-\eta_k) \to \int_{\Omega\times\{t\}} A_{\mathcal{N}}(v)(\nabla v, \nabla v)\psi,\end{aligned}$$

and

$$\Big|\int_{\Omega} \langle \nabla v, \psi\nabla\eta_k\rangle\Big| \le C\Big(\int_{\Omega} |\nabla v|^2\Big)^{\frac12}\Big(\int_{\Omega} |\nabla\eta_k|^2\Big)^{\frac12} \to 0.$$

With these estimates we obtain

$$\int_{\Omega\times\{t\}} \langle \tau(\cdot,t), D\Pi_{\mathcal{N}}(v)\psi\rangle = \int_{\Omega\times\{t\}} \langle \nabla v, \nabla\psi\rangle - A_{\mathcal{N}}(v)(\nabla v, \nabla v)\psi \tag{3.13}$$

for any $t \in A$ and $\psi \in C_0^{\infty}(\Omega, \mathbb{R}^L)$.

On other hand, by comparing (3.3) with (3.11), we see that for a.e. $t \in A$,

$$\tau(\cdot,t) = (v_t + u\cdot\nabla v)(\cdot,t).$$

Since $\tau(\cdot,t) \in T_{v(\cdot,t)}\mathcal{N}$ holds a.e. in Ω, we have that

$$\int_{\Omega} \langle \tau(t), D\Pi_{\mathcal{N}}(v)\phi\rangle = \int_{\Omega} \langle \tau(t), \phi\rangle = \int_{\Omega} \langle v_t + u\cdot\nabla v, \phi\rangle.$$

This, combined with (3.13), implies that

$$v_t + u\cdot\nabla v - \Delta v = A_{\mathcal{N}}(v)(\nabla v, \nabla v) \tag{3.14}$$

holds weakly in Ω for a.e. $t \in A$. Hence $(1.1)_3$ holds.

Next we proceed to verify the difficult part, that is, u solves $(1.1)_1$. First by the estimate (3.2), we have

$$u_t^{\varepsilon} \rightharpoonup u_t, \quad \text{in } L^2([0,T], H^{-1}) \cap L^2([0,T], W^{-2,p})$$

for some $1 < p < 2$. For any $\xi \in C^{\infty}([0,T])$ with $\xi(T) = 0$, $\varphi \in \mathbf{J}$, since

$$\int_{Q_T} u_t^{\varepsilon}\xi\varphi = -\int_{\Omega} u_0\xi(0)\varphi - \int_{Q_T} u^{\varepsilon}\xi'\varphi,$$

this, after taking $\epsilon \to 0$, implies that

$$\int_{Q_T} u_t \xi \varphi = -\int_\Omega u_0 \xi(0)\varphi - \int_{Q_T} u\xi' \varphi.$$

By the definition of A, it is readily seen that for any $t \in A$, it holds

$$0 = \int_{\Omega\times\{t\}} \langle \partial_t u^\varepsilon, \varphi\rangle + \int_{\Omega\times\{t\}} \langle u^\varepsilon \cdot \nabla u^\varepsilon, \varphi\rangle + \int_{\Omega\times\{t\}} \langle \nabla u^\varepsilon, \nabla\varphi\rangle + \int_{\Omega\times\{t\}} (\nabla v^\varepsilon \odot \nabla v^\varepsilon) : \nabla\varphi, \tag{3.15}$$

for any $\varphi \in \mathbf{J}$.

Now we need to show the following crucial claim.

Claim: For any $t \in A$, it holds that for any $\varphi \in \mathbf{J}$,

$$\begin{aligned}&\int_{\Omega\times\{t\}} \langle \partial_t u^\varepsilon, \varphi\rangle + \int_{\Omega\times\{t\}} \langle u^\varepsilon \cdot \nabla u^\varepsilon, \varphi\rangle + \int_{\Omega\times\{t\}} \langle \nabla u^\varepsilon, \nabla\varphi\rangle + \int_{\Omega\times\{t\}} (\nabla v^\varepsilon \odot \nabla v^\varepsilon) : \nabla\varphi \\ &\to \int_{\Omega\times\{t\}} \langle u_t, \varphi\rangle + \int_{\Omega\times\{t\}} \langle u \cdot \nabla u, \varphi\rangle + \int_{\Omega\times\{t\}} \langle \nabla u, \nabla\varphi\rangle + \int_{\Omega\times\{t\}} (\nabla v \odot \nabla v) : \nabla\varphi,\end{aligned} \tag{3.16}$$

as $\varepsilon \to 0$.

Observe by (3.7) that for any $t \in A$, $\nabla u^\varepsilon(\cdot, t) \rightharpoonup \nabla u(\cdot, t)$ in $L^2(\Omega)$. Thus the convergence of the first three terms follows immediately.

The crucial step of this claim is to show the weak convergence of Ericksen stress tensors of v^ε to that of v, i.e.,

$$\lim_{\varepsilon\to 0} \int_{\Omega\times\{t\}} (\nabla v^\varepsilon \odot \nabla v^\varepsilon) : \nabla\varphi = \int_{\Omega\times\{t\}} (\nabla v \odot \nabla v) : \nabla\varphi, \ \forall \varphi \in \mathbf{J}. \tag{3.17}$$

For simplicity, we may assume $\Sigma_t = \{(0, 0)\} \subset \Omega$ consists of a single point at zero. Let $\varphi \in C_0^\infty(\Omega, \mathbb{R}^2)$ be such that $div\phi = 0$ and $(0, 0) \in \mathrm{supp}(\varphi)$, the support of φ. Observe that

$$\int_{\Omega\times\{t\}} (\nabla v^\varepsilon \odot \nabla v^\varepsilon) : \nabla\varphi = \int_{\Omega\times\{t\}} \left(\nabla v^\varepsilon \odot \nabla v^\varepsilon - \frac{1}{2}|\nabla v^\varepsilon|^2 \mathbb{I}_2\right) : \nabla\varphi.$$

Also by direct computations we observe that

$$\nabla v^\varepsilon \odot \nabla v^\varepsilon - \frac{1}{2}|\nabla v^\varepsilon|^2 \mathbb{I}_2 = \frac{1}{2}\begin{pmatrix} |\partial_x v^\varepsilon|^2 - |\partial_y v^\varepsilon|^2, & 2\langle \partial_x v^\varepsilon, \partial_y v^\varepsilon\rangle \\ 2\langle \partial_x v^\varepsilon, \partial_y v^\varepsilon\rangle, & |\partial_y v^\varepsilon|^2 - |\partial_x v^\varepsilon|^2 \end{pmatrix} \tag{3.18}$$

is a 2×2-matrix valued function whose entries consist of the Hopf differential of v^ε. It follows from (3.4) and (2.27) of Lemma 2.2 that there are two real numbers α, β such that

$$(|\partial_x v^\varepsilon|^2 - |\partial_y v^\varepsilon|^2) d\mathbf{x} \rightharpoonup (|\partial_x v|^2 - |\partial_y v|^2) d\mathbf{x} + \alpha\delta_{(0,0)}, \tag{3.19}$$

and

$$\langle \partial_x v^\varepsilon, \partial_y v^\varepsilon\rangle d\mathbf{x} \rightharpoonup \langle \partial_x v, \partial_y v\rangle d\mathbf{x} + \beta\delta_{(0,0)}, \tag{3.20}$$

as convergence of Radon measures.

The above claim will follow if we can show a stronger property, namely the weak convergence of the Ericksen stress tensors in (3.19) and (3.20):

$$\alpha = \beta = 0. \tag{3.21}$$

Recall that

$$\Delta v^\varepsilon - \frac{1}{\varepsilon^2}\chi'\left(\mathrm{dist}^2(v^\varepsilon, \mathcal{N})\right)\frac{d}{dv}\left(\mathrm{dist}^2(v^\varepsilon, \mathcal{N})\right) = \mathbf{f}^\varepsilon := \partial_t v^\varepsilon + u^\varepsilon \cdot \nabla v^\varepsilon, \tag{3.22}$$

where the tension filed $\mathbf{f}^\varepsilon$ is uniformly bounded in $L^2(\Omega)$. Denote the Ginzburg-Landau energy density by

$$e_\varepsilon(v^\varepsilon) = \frac{1}{2}|\nabla v^\varepsilon|^2 + \frac{1}{\varepsilon^2}\chi\left(\mathrm{dist}^2(v^\varepsilon, \mathcal{N})\right).$$

Next we want to derive the Pohozaev identity for v^ε. For any $X \in C_0^\infty(\Omega, \mathbb{R}^2)$, by multiplying the equation (3.22) by $X \cdot \nabla v^\varepsilon$ and integrating over $B_r(0)$ we get

$$\begin{aligned}\int_{\partial B_r(0)}(X^j v_j^\varepsilon)\cdot\left(v_i^\varepsilon\frac{\mathbf{x}^i}{|\mathbf{x}|}\right) - \int_{B_r(0)} X_i^j v_j^\varepsilon \cdot v_i^\varepsilon + \int_{B_r(0)} \mathrm{div} X e_\varepsilon(v^\varepsilon) - \int_{\partial B_r(0)} e_\varepsilon(v^\varepsilon)\left(X\cdot\frac{\mathbf{x}}{|\mathbf{x}|}\right)\\ = \int_{B_r(0)}(X\cdot\nabla v^\varepsilon)\cdot\mathbf{f}^\varepsilon.\end{aligned} \tag{3.23}$$

If we choose $X(\mathbf{x}) = \mathbf{x}$, then it holds that

$$r\int_{\partial B_r(0)}\left|\frac{\partial v^\varepsilon}{\partial r}\right|^2 + \int_{B_r(0)}\frac{2}{\varepsilon^2}\chi(\mathrm{dist}^2(v^\varepsilon, \mathcal{N})) - r\int_{\partial B_r(0)} e_\varepsilon(v^\varepsilon) = \int_{B_r(0)}|\mathbf{x}|\frac{\partial v^\varepsilon}{\partial r}\cdot\mathbf{f}^\varepsilon, \tag{3.24}$$

and hence

$$\int_{\partial B_r(0)} e_\varepsilon(v^\varepsilon) = \int_{\partial B_r(0)}\left|\frac{\partial v^\varepsilon}{\partial r}\right|^2 + \frac{1}{r}\int_{B_r(0)}\frac{2}{\varepsilon^2}\chi(\mathrm{dist}^2(v^\varepsilon, \mathcal{N})) + O\Big(\int_{B_r(0)}|\nabla v^\varepsilon||\mathbf{f}^\varepsilon|\Big). \tag{3.25}$$

This, after integrating from r to R, yields that

$$\begin{aligned}\int_{B_R(0)\setminus B_r(0)} e_\varepsilon(v^\varepsilon) = \int_{B_R(0)\setminus B_r(0)}\left|\frac{\partial v^\varepsilon}{\partial r}\right|^2 + \int_r^R \frac{1}{\tau}\int_{B_\tau(0)}\frac{2}{\varepsilon^2}\chi(\mathrm{dist}^2(v^\varepsilon, \mathcal{N}))d\tau\\ + \int_r^R O\Big(\int_{B_\tau(0)}|\nabla v^\varepsilon||\mathbf{f}^\varepsilon|\Big)d\tau.\end{aligned} \tag{3.26}$$

Since $\Sigma_t = \{(0,0)\}$, we can assume that there exists $\gamma \geq 0$ such that

$$e_\varepsilon(v^\varepsilon)d\mathbf{x} \rightharpoonup \frac{1}{2}|\nabla v|^2 d\mathbf{x} + \gamma\delta_{(0,0)} \quad \text{in} \quad B_\delta(0), \tag{3.27}$$

as convergence of Radon measures. For $t \in A$, we have that

$$\lim_{\varepsilon\to 0}\int_{B_\tau(0)}|\mathbf{f}^\varepsilon||\nabla v^\varepsilon| \leq \lim_{\varepsilon\to 0}\Big(\int_{B_\tau(0)}|\mathbf{f}^\varepsilon|^2\Big)^{\frac{1}{2}}\Big(\int_{B_\tau(0)}|\nabla v^\varepsilon|^2\Big)^{\frac{1}{2}} \leq CE_0.$$

Hence, after sending $\varepsilon \to 0$, we obtain from (3.26) that

$$\int_{B_R(0)\setminus B_r(0)}\frac{1}{2}|\nabla v|^2 \geq \int_{B_R(0)\setminus B_r(0)}\left|\frac{\partial v}{\partial r}\right|^2 + \int_r^R \frac{1}{\tau}\lim_{\varepsilon\to 0}\int_{B_\tau(0)}\frac{2}{\varepsilon^2}\chi(\mathrm{dist}^2(v^\varepsilon, \mathcal{N}))d\tau + O(R).$$

Sending $r \to 0$, this further implies that

$$\int_{B_R(0)} \frac{1}{2}|\nabla v|^2 \geq \int_{B_R(0)} \left|\frac{\partial v}{\partial r}\right|^2 + \int_0^R \frac{1}{\tau} \lim_{\varepsilon\to 0} \int_{B_\tau(0)} \frac{2}{\varepsilon^2}\chi(\mathrm{dist}^2(v^\varepsilon, \mathcal{N}))d\tau + O(R).$$

From this, we must have that

$$\frac{2}{\varepsilon^2}\chi(\mathrm{dist}^2(v^\varepsilon, \mathcal{N})) \to 0 \quad \text{in} \quad L^1(B_\delta). \tag{3.28}$$

For, otherwise, we would have that

$$\frac{2}{\varepsilon^2}\chi(\mathrm{dist}^2(v^\varepsilon, \mathcal{N}))\, dx \rightharpoonup \kappa\delta_{(0,0)}$$

for some $\kappa > 0$ so that

$$\int_0^R \frac{1}{\tau} \lim_{\varepsilon\to 0} \int_{B_\tau(0)} \frac{2}{\varepsilon^2}\chi(\mathrm{dist}^2(v^\varepsilon, \mathcal{N})) = \int_0^R \frac{\kappa}{\tau} d\tau = \infty.$$

This is impossible.

Next, by choosing $X(\mathbf{x}) = (x, 0)$ in (3.23) we obtain that

$$\begin{aligned} &\frac{1}{2}\int_{B_r(0)} \left(\left|\partial_y v^\varepsilon\right|^2 - \left|\partial_x v^\varepsilon\right|^2\right) + \int_{B_r(0)} \frac{1}{\varepsilon^2}\chi(\mathrm{dist}^2(v^\varepsilon, \mathcal{N})) \\ &= \int_{B_r(0)} x\langle \partial_x v^\varepsilon, \mathbf{f}^\varepsilon\rangle + \int_{\partial B_r(0)} \frac{x^2}{r} e_\varepsilon(v^\varepsilon) - \int_{\partial B_r(0)} x\langle \partial_x v^\varepsilon, \frac{\partial v^\varepsilon}{\partial r}\rangle. \end{aligned} \tag{3.29}$$

Observe that by Fubini's theorem, for a.e. $r > 0$ it holds that

$$\int_{\partial B_r(0)} x\langle \partial_x v^\varepsilon, \frac{\partial v^\varepsilon}{\partial r}\rangle \to \int_{\partial B_r(0)} x\langle \partial_x v, \frac{\partial v}{\partial r}\rangle,$$

$$\int_{\partial B_r(0)} \frac{x^2}{r} e_\varepsilon(v^\varepsilon) \to \frac{1}{2}\int_{\partial B_r(0)} \frac{x^2}{r}|\nabla v|^2,$$

and by (3.28),

$$\int_{B_r(0)} \frac{1}{\varepsilon^2}\chi(\mathrm{dist}^2(v^\varepsilon, \mathcal{N})) \to 0.$$

Furthermore,

$$\left|\int_{B_r(0)} x\langle \partial_x v^\varepsilon, \mathbf{f}^\varepsilon\rangle\right| \leq Cr \left\|\mathbf{f}^\varepsilon\right\|_{L^2} \left\|\nabla v^\varepsilon\right\|_{L^2} = O(r).$$

Hence, by sending $\varepsilon \to 0$ in (3.29), we obtain that

$$\int_{B_r(0)} \left(\left|\partial_y v\right|^2 - \left|\partial_x v\right|^2\right) + \alpha = O(r).$$

This implies that $\alpha = 0$, after sending $r \to 0$.

Similarly, if we choose $X(\mathbf{x}) = (0, x)$ in (3.23) and pass the limit in the resulting equation, we can get that

$$\int_{B_r(0)} \left\langle \partial_x v, \partial_y v\right\rangle + \beta = O(r).$$

This can imply that $\beta = 0$ after sending $r \to 0$. This proves (3.21) and (3.18). Hence the Claim holds.

Finally, by multiplying (3.16) by $\xi \in C^\infty([0, T])$ with $\xi(T) = 0$ and integrating over $[0, T]$, we conclude that u satisfies the $(1.1)_1$ on Q_T. The proof of Theorem 1.2 is complete.

4 Compactness of weak solutions to the simplified Ericksen-Leslie system

This section is devoted to the proof of Theorem 1.3. A key ingredient is the L^p-estimate, $1 < p < 2$, for the Hopf differential of v^k.

Since (u^k, v^k) satisfies the assumption (1.9), and

$$(u_0^k, v_0^k) \rightharpoonup (u_0, v_0) \quad \text{in} \quad L^2(\Omega) \times H^1(\Omega),$$

there exists $(u(\mathbf{x}, t), v(\mathbf{x}, t)) : Q_T \to \mathbb{R}^2 \times \mathcal{N}$ such that

$$(u^k, v^k) \rightharpoonup (u, v) \ \text{ in } \ L^2([0, T], H^1(\Omega)), \tag{4.1}$$

$$v_t^k + u^k \cdot \nabla v^k \rightharpoonup v_t + u \cdot \nabla v \ \text{ in } \ L^2([0, T], L^2(\Omega)). \tag{4.2}$$

Also it follows from the standard estimates on the system (1.1) and (1.9), similar to the discussion we have in the previous section, that there exists $1 < q < 2$ such that

$$\sup_k \left[\|u_t^k\|_{L_t^2 W_x^{-2,q}} + \|v_t^k\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}} \right] < \infty. \tag{4.3}$$

Hence, by the Aubin-Lions Lemma we may assume that

$$(u^k, v^k) \to (u, v) \ \text{ in } \ L^2(Q_T) \times L^2(Q_T),$$

and

$$(u^k(\cdot, t), v^k(\cdot, t)) \rightharpoonup (u(\cdot, t), v(\cdot, t)) \text{ in } L^2(\Omega)$$

for all $t \in [0, T]$.

By the lower semi-continuity, we have

$$\int_{Q_t} \left(|\nabla u|^2 + |v_t + u \cdot \nabla v|^2\right) \leq \liminf_{k\to\infty} \int_{Q_t} \left(|\nabla u^k|^2 + |v_t^k + u^k \cdot \nabla v^k|^2\right) \leq C_0.$$

By Fatou's Lemma and (1.9), we have

$$\int_0^t \liminf_{k\to\infty} \int_\Omega \left(|\nabla u^k|^2 + |v_t^k + u^k \cdot \nabla v^k|^2\right) \leq \liminf_{k\to\infty} \int_{Q_t} \left(|\nabla u^k|^2 + |v_t^k + u^k \cdot \nabla v^k|^2\right) \leq C_0.$$

Hence, there exists $A \subset [0, T]$ with full Lebesgue measure T such that for all $t \in A$

$$\liminf_{k\to\infty} \int_\Omega \left(|\nabla u^k|^2 + |v_t^k + u^k \cdot \nabla v^k|^2\right)(t) < \infty. \tag{4.4}$$

Now we define the concentration set at time $t \in A$ by letting

$$\Sigma_t := \bigcap_{r>0} \left\{ x \in \Omega : \liminf_{k\to\infty} \int_{B_r(x)} |\nabla v^k|^2 > \delta_0^2 \right\}, \tag{4.5}$$

where δ_0 is the same constant as in Theorem 1.2 in [28]. As in [28] (see also [24, 31]), we can show that for any $t \in A$, it holds that $\#(\Sigma_t) \leq C(E_0)$ and

$$v^k(t) \to v \quad \text{in} \quad H^1_{\mathrm{loc}}(\Omega \setminus \Sigma_t). \tag{4.6}$$

Similar to the proof of Theorem 1.2, we can show the weak limit (u, v) satisfies the third equation of (1.1) in the weak sense. The most difficult part is to show that the first equation of (1.1) also holds in the weak sense. As in the proof of Theorem 1.2, in order to complete the proof of Theorem 1.3, it is suffices to show the following convergence of Ericksen stress tensors:

$$\lim_{k\to\infty}\int_{\Omega\times\{t\}}\left(\nabla v^k\odot\nabla v^k\right):\nabla\varphi=\int_{\Omega\times\{t\}}(\nabla v\odot\nabla v):\nabla\varphi,\ \forall\varphi\in\mathbf{J}. \tag{4.7}$$

For simplicity, assume $\Sigma_t=\{(0,0)\}\subset\Omega$. Let $\varphi\in C^\infty(\Omega,\mathbb{R}^2)$ be such that $\operatorname{div}\varphi=0$ and $(0,0)\in\operatorname{spt}(\varphi)$. By the same calculation as in (3.18), we have

$$\nabla v^\varepsilon\odot\nabla v^\varepsilon-\frac{1}{2}|\nabla v^\varepsilon|^2\mathbb{I}_2=\frac{1}{2}\begin{pmatrix}|\partial_x v^\varepsilon|^2-|\partial_y v^\varepsilon|^2, & 2\langle\partial_x v^\varepsilon,\partial_y v^\varepsilon\rangle\\ 2\langle\partial_x v^\varepsilon,\partial_y v^\varepsilon\rangle, & |\partial_y v^\varepsilon|^2-|\partial_x v^\varepsilon|^2\end{pmatrix}.$$

For any $t\in A$, note that $v^k(t)$ is an approximated harmonic maps from Ω to $\mathcal{N}$:

$$\Delta v^k+A(v^k)(\nabla v^k,\nabla v^k)=g^k(t):=v_t^k(t)+u^k\cdot\nabla v^k(t)\in L^2(\Omega). \tag{4.8}$$

By higher order Sobolev regularity of approximated harmonic maps with L^2-tension fields in dimension two, see [28] and [31], we have $v^k\in W^{2,2}(\Omega,\mathcal{N})$.

Recall the Hopf differential of v^k is defined by

$$\mathcal{H}^k=\left(\frac{\partial v^k}{\partial z}\right)^2=\left|\partial_x v^k\right|^2-\left|\partial_y v^k\right|^2+2i\left\langle\partial_x v^k,\partial_y v^k\right\rangle, \tag{4.9}$$

where $z=x+iy\in\mathbb{C}$. Since $v^k\in W^{2,2}(\Omega,\mathcal{N})$, direct calculations give

$$\frac{\partial\mathcal{H}^k}{\partial\bar z}=2\frac{\partial v^k}{\partial z}\frac{\partial^2 v^k}{\partial\bar z\partial z}=2\Delta v^k\frac{\partial v^k}{\partial z}=2g^k(t)\frac{\partial v^k}{\partial z}:=G^k. \tag{4.10}$$

It is clear that

$$\|G^k\|_{L^1(B_r)}\le 2\|g^k(t)\|_{L^2(\Omega)}\left\|\frac{\partial v^k}{\partial z}\right\|_{L^2(\Omega)}\le 2C_0. \tag{4.11}$$

Therefore, for any $z\in B_r(0)$ we have that

$$\mathcal{H}^k(z)=\int_{\partial B_{2r}(0)}\frac{\mathcal{H}^k(\omega)}{z-\omega}\,d\sigma+\int_{B_{2r}(0)}\frac{G^k(\omega)}{z-\omega}\,d\omega. \tag{4.12}$$

By the Young inequality of convolutions, we obtain that

$$\|\mathcal{H}^k\|_{L^p(B_r)}\le C(r,p)\|\mathcal{H}^k\|_{L^1(\partial B_{2r})}+\left\|\frac{1}{z}\right\|_{L^p}\|G^k\|_{L^1(B_{2r})}\le C(r,p) \tag{4.13}$$

holds for any $1<p<2$. From this and the convergence $\nabla v^k\to\nabla v$ in $L^2_{\text{loc}}(\Omega\setminus\Sigma_t,\mathcal{N})$, we immediately conclude that

$$|\partial_x v^k|^2-|\partial_y v^k|^2\rightharpoonup|\partial_x v|^2-|\partial_y v|^2,\quad\langle\partial_x v^k,\partial_y v^k\rangle\rightharpoonup\langle\partial_x v,\partial_y v\rangle\quad\text{in}\quad L^p(B_r(0))$$

holds for any $1<p<2$, which implies the L^1-weak convergence of Ericksen stress tensors of v^k to that of v. In particular, (4.7) holds true. This completes the proof of Theorem 1.3.

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