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# *Suitable Weak Solutions for the Co-rotational Beris–Edwards System in Dimension Three*

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**Abstract**

In this paper, we establish the global existence of a suitable weak solution to the co-rotational Beris–Edwards $Q$-tensor system modeling the hydrodynamic motion of nematic liquid crystals with either Landau–De Gennes bulk potential in $\mathbb{R}^3$ or Ball–Majumdar bulk potential in $\mathbb{T}^3$, a system coupling the forced incompressible Navier–Stokes equation with a dissipative, parabolic system of $Q$-tensor $Q$ in $\mathbb{R}^3$, which is shown to be smooth away from a closed set $\Sigma$ whose 1-dimensional parabolic Hausdorff measure is zero.

## 1. Introduction

In this paper, we consider in dimension three the so-called Beris–Edwards system [4,10] that describes the hydrodynamic motion of nematic liquid crystals, with either the Landau–De Gennes bulk potential function [8] or the Maire–Saupe (Ball–Majumdar) bulk potential function [3]. Roughly speaking, this is a system that couples a forced Navier–Stokes equation for the underlying fluid velocity field $u$ with a dissipative parabolic system of $Q$-tensors modeling nematic liquid crystal orientation fields. We are interested in establishing the existence of certain global weak solutions for such a Beris–Edwards system that enjoys partial smoothness property, analogous to the celebrated works by Cafferalli–Kohn–Nirenberg [5] on the Navier–Stokes equation and Lin-Liu [24] and [25] on the simplified Ericksen–Leslie system modeling nematic liquid crystal flows with variable degree of orientations, which was proposed by Ericksen [12,13] and Leslie [22] in the 1960's.

We begin with the description of this system. Recall that the configuration space of $Q$-tensors is the set of traceless, symmetric $3 \times 3$-matrices, i.e.,

$$\mathcal{S}_0^{(3)} = \left\{ Q \in \mathbb{R}^{3\times 3} :\ Q = Q^\top,\ \mathrm{tr}\, Q = 0 \right\}.$$

For technical reasons, we will consider the one constant approximate form of the Landau–De Gennes energy functional of $Q$-tensors, namely,

$$E(Q) = \int_\Omega \Big(\frac{L}{2}|\nabla Q|^2 + F_{\text{bulk}}(Q)\Big)\,\mathrm{d}x,$$

over the Sobolev space $H^1(\Omega, \mathcal{S}_0^{(3)})$, where $\Omega$ is a three dimensional domain that is assumed to be either $\mathbb{R}^3$ or the torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ in this paper. Here $L > 0$ denotes the elasticity constant, and $F_{\text{bulk}}(Q)$ denotes the bulk potential function that usually describes the phase transition among various phase states including isotropic, uniaxial, or biaxial states. We refer interested readers to Mottram-Newton [29] and Sonnet–Virga [33] for a more detailed discussion of general Landau–De Gennes energy functionals involving multiple elasticity constants $L_i$'s. In this paper, we will consider two classes of bulk potential functions:

(i) (Landau–De Gennes bulk potential [8].) Here $F_{\text{bulk}}(Q) = F_{\text{LdG}}(Q)$, and

$$F_{\text{LdG}}(Q) = \widehat{F}_{\text{LdG}}(Q) - \min_{Q' \in \mathcal{S}_0^{(3)}} \widehat{F}_{\text{LdG}}(Q'), \tag{1.1}$$

where

$$\widehat{F}_{\text{LdG}}(Q) = \frac{a}{2}\mathrm{tr}(Q^2) - \frac{b}{3}\mathrm{tr}(Q^3) + \frac{c}{4}\mathrm{tr}^2(Q^2), \tag{1.2}$$

where $a, b, c > 0$ are temperature dependent material constants. It is a well known fact that if $0 < a < \frac{b^2}{27c}$, then $\widehat{F}_{\text{LdG}}$ reaches its minimum at $Q = s_+(d \otimes d - \frac{1}{3}I_3)$, where $s_+ = \frac{b+\sqrt{b^2-24ac}}{4c}$ and $d \in \mathbb{S}^2$ is a unit vector field.

(ii) (Ball–Majumdar singular bulk potential [3].) Here $F_{\text{bulk}}(Q) = F_{\text{BM}}(Q)$ is a modified Maire-Saupe bulk potential introduced by Ball–Majumdar [3], which is defined as follows. $F_{\text{BM}}(Q) = G_{\text{BM}}(Q) - \frac{\kappa}{2}|Q|^2$ for some $\kappa > 0$, and

$$G_{\text{BM}}(Q) \equiv \begin{cases} \min\limits_{\rho \in \mathcal{A}_Q} \displaystyle\int_{\mathbb{S}^2} \rho(p)\log\rho(p)\,\mathrm{d}\sigma(p) & \text{if } -\frac{1}{3} < \lambda_j(Q) < \frac{2}{3}, \\ \infty & \text{otherwise}, \end{cases} \tag{1.3}$$

where $\lambda_j$, $j = 1, 2, 3$, denotes the eigenvalues of $Q \in \mathcal{S}_0^{(3)}$, and

$$\mathcal{A}_Q \equiv \Big\{0 \le \rho \in L^1(\mathbb{S}^2) : \rho(p) = \rho(-p), \int_{\mathbb{S}^2} \rho(p)\,\mathrm{d}\sigma(p) = 1, \\ \int_{\mathbb{S}^2} \big(p \otimes p - \frac{1}{3}I_3\big)\rho(p)\,\mathrm{d}\sigma(p) = Q\Big\}.$$

It was proven by [3] that $G_{\text{BM}}$ is strictly convex and smooth in the interior of the convex set

$$\mathcal{D} = \Big\{Q \in \mathcal{S}_0^{(3)} : -\frac{1}{3} \le \lambda_i(Q) \le \frac{2}{3},\ i = 1, 2, 3\Big\}.$$

It is well-known that the first order variation of the Landau–De Gennes energy functional $E$ is given by

$$H = L\Delta Q - f_{\text{bulk}}(Q), \quad f_{\text{bulk}}(Q) = \langle \nabla F_{\text{bulk}}(Q)\rangle = \nabla F_{\text{bulk}}(Q) - \frac{\text{tr}(\nabla F_{\text{bulk}}(Q))}{3} I_3. \tag{1.4}$$

In particular, if $F_{\text{bulk}}(Q) = F_{\text{LdG}}(Q)$, then

$$f_{\text{bulk}}(Q) = \langle \nabla F_{\text{LdG}}(Q)\rangle = aQ - b\big[Q^2 - \frac{\text{tr}(Q^2)}{3} I_3\big] + cQ\text{tr}(Q^2).$$

For $0 < T \le \infty$, denote $Q_T = \Omega \times (0, T]$. Let $\mathbf{u} : Q_T \mapsto \mathbb{R}^3$ denote the fluid velocity field and $Q : Q_T \mapsto \mathcal{S}_0^{(3)}$ denote the director field. Define

$$S(\nabla \mathbf{u}, Q) = (\xi D + \omega)\big(Q + \frac{1}{3} I_3\big) + \big(Q + \frac{1}{3} I_3\big)(\xi D - \omega) - 2\xi\big(Q + \frac{1}{3} I_3\big)\text{tr}(Q\nabla \mathbf{u}),$$

where

$$D = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^{\top}) \text{ and } \omega = \frac{1}{2}(\nabla \mathbf{u} - (\nabla \mathbf{u})^{\top})$$

are the symmetric part and the antisymmetric part, respectively, of the velocity gradient tensor $\nabla \mathbf{u}$, and $\xi \in \mathbb{R}$ is a rotational parameter measuring the ratio between the aligning and tumbling effects to $Q$ by the fluid velocity field.

The Beris–Edwards $Q$-tensor system modeling the hydrodynamic motion of nematic liquid crystals reads as [15,30]

$$\begin{cases} \partial_t Q + \mathbf{u} \cdot \nabla Q - S(\nabla \mathbf{u}, Q) = \Gamma H \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \mu \Delta \mathbf{u} + \text{div}(\tau + \sigma) \\ \text{div} \mathbf{u} = 0, \end{cases} \tag{1.5}$$

where $\Gamma > 0$ is a relaxation time parameter, $\mu > 0$ is the fluid viscosity constant, and $\tau$ is the symmetric part of the additional stress tensor given by

$$\tau_{\alpha\beta} = -\xi\big(Q_{\alpha\gamma} + \frac{\delta_{\alpha\gamma}}{3}\big)H_{\gamma\beta} - \xi H_{\alpha\gamma}\big(Q_{\gamma\beta} + \frac{\delta_{\gamma\beta}}{3}\big) + 2\xi\big(Q_{\alpha\beta} + \frac{\delta_{\alpha\beta}}{3}\big)Q_{\gamma\delta}H_{\gamma\delta} - L\partial_\beta Q_{\gamma\delta}\partial_\alpha Q_{\gamma\delta}, \ 1 \le \alpha, \beta \le 3,$$

and $\sigma$ is the antisymmetric part of the additional stress tensor:

$$\sigma_{\alpha\beta} = [Q, H]_{\alpha\beta} := Q_{\alpha\gamma}H_{\gamma\beta} - H_{\alpha\gamma}Q_{\gamma\beta}, \ 1 \le \alpha, \beta \le 3.$$

Since both $f_{\text{LdG}}(Q)$ and $f_{\text{BM}}(Q)$ are isotropic functions of $Q$, we have

$$[Q, f_{\text{bluk}}(Q)] = 0,$$

so that

$$\sigma = [Q, L\Delta Q - f_{\text{bulk}}(Q)] = L[Q, \Delta Q].$$

In this paper, we will focus on the co-rotational Beris–Edwards system (1.5), i.e.,

$$\boxed{\xi = 0}.$$

Since the exact values of $L$, $\Gamma$, $\mu$ don't play roles in our analysis, we will assume, for simplicity,

$$\boxed{L = \Gamma = \mu = 1}.$$

We will also assume the domain $\Omega$ to be

$$\boxed{\Omega = \begin{cases} \mathbb{R}^3 & if \;\; F_{\text{bulk}}(Q) = F_{\text{LdG}}(Q), \\ \mathbb{T}^3 & if \;\; F_{\text{bulk}}(Q) = F_{\text{BM}}(Q). \end{cases}}$$

With these assumptions and the identity

$$\partial_\beta(\partial_\beta Q_{\gamma\delta}\partial_\alpha Q_{\gamma\delta}) = \partial_\alpha Q_{\gamma\delta}\Delta Q_{\gamma\delta} + \partial_\alpha\left(\frac{1}{2}|\nabla Q|^2\right),$$

the system (1.5) reduces to the following form:

$$\begin{cases} \partial_t Q + \mathbf{u}\cdot\nabla Q - [\omega, Q] = \Delta Q - f_{\text{bulk}}(Q), \\ \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla P = \Delta\mathbf{u} - \nabla Q\cdot\Delta Q + \operatorname{div}[Q, \Delta Q], \;\; \text{in } \Omega\times(0,\infty)\;, \\ \operatorname{div}\mathbf{u} = 0, \end{cases} \tag{1.6}$$

subject to the initial condition

$$(\mathbf{u}, Q)|_{t=0} = (\mathbf{u}_0, Q_0)(x) \quad \text{for} \quad x\in\Omega. \tag{1.7}$$

A key feature of the Beris–Edwards system (1.6) (or (1.5) in general) is the energy dissipation property, which plays a fundamental role in the analysis of (1.6). More precisely, if $(\mathbf{u}, Q) : \Omega\times(0,\infty) \mapsto \mathbb{R}^3\times\mathcal{S}_0^{(3)}$ is a sufficiently regular solution of (1.5), then it satisfies the following energy inequality [30,31]:

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\mathbf{u}, Q)(t) = -\int_\Omega (|\nabla\mathbf{u}|^2 + |H|^2)(x,t)\,\mathrm{d}x, \tag{1.8}$$

where

$$E(\mathbf{u}, Q)(t) = \int_\Omega \left(\frac{1}{2}|\mathbf{u}|^2 + \frac{1}{2}|\nabla Q|^2 + F_{\text{bulk}}(Q)\right)(x,t)\,\mathrm{d}x \tag{1.9}$$

is the total energy of the complex fluid consisting of the elastic energy of the director field $Q$ and the kinetic energy of the underlying fluid $\mathbf{u}$. While the right hand side of (1.8) denotes the dissipation rate of this system of complex fluid.

*Some Notations* For $Q \in \mathcal{S}_0^{(3)}$, we use the Frobenius norm of $Q$, i.e.

$$|Q| = \sqrt{\mathrm{tr}(Q^2)} = \sqrt{Q_{\alpha\beta}Q_{\alpha\beta}},$$

and the Sobolev spaces of $Q$-tensors, $W^{l,p}(\Omega, \mathcal{S}_0^{(3)})$ ($l \in \mathbb{N}_+$ and $1 \le p \le \infty$), are defined by

$$W^{l,p}(\Omega, \mathcal{S}_0^{(3)}) = \left\{ Q = (Q_{\alpha\beta}) : \Omega \mapsto \mathcal{S}_0^{(3)} : \ Q_{\alpha\beta} \in W^{l,p}(\Omega), \ \forall 1 \le \alpha, \beta \le 3 \right\}.$$

When $p = 2$, we denote $W^{l,2}(\Omega, \mathcal{S}_0^{(3)})$ by $H^l(\Omega, \mathcal{S}_0^{(3)})$. For $A, B \in \mathbb{R}^{3\times 3}$, we denote

$$A : B = A_{\alpha\beta}B_{\alpha\beta}, \ \ A \cdot B = \mathrm{tr}(AB), \ \ |\nabla Q|^2 = Q_{\alpha\beta,\gamma}Q_{\alpha\beta,\gamma}, \ \ |\Delta Q|^2 = \Delta Q_{\alpha\beta}\Delta Q_{\alpha\beta},$$

and

$$(\mathbf{u} \otimes \mathbf{u})_{\alpha\beta} = \mathbf{u}_\alpha \mathbf{u}_\beta, \ \ (\nabla Q \otimes \nabla Q)_{\alpha\beta} = \nabla_\alpha Q_{\gamma\delta} \nabla_\beta Q_{\gamma\delta}.$$

Note that $A : B = A \cdot B$ for $A, B \in \mathcal{S}_0^{(3)}$. We also use $A_{\mathrm{sym}}, A_{\mathrm{anti}}$ to denote the symmetric and antisymmetric parts of $A$, respectively.

Define

$$\mathbf{H} = \text{Closure of } \left\{ \mathbf{u} \in C_0^\infty(\Omega, \mathbb{R}^3) : \ \mathrm{div}\mathbf{u} = 0 \right\} \text{ in } L^2(\Omega),$$

and

$$\mathbf{V} = \text{Closure of } \left\{ \mathbf{u} \in C_0^\infty(\Omega, \mathbb{R}^3) : \ \mathrm{div}\mathbf{u} = 0 \right\} \text{ in } H^1(\Omega).$$

For $0 \le k \le 5$, $\mathcal{P}^k$ denotes the $k$-dimensional Hausdorff measure on $\mathbb{R}^3 \times \mathbb{R}_+$ with respect to the parabolic distance:

$$\delta((x,t),(y,s)) = \max\left\{ |x-y|, \sqrt{|t-s|} \right\}, \ \forall (x,t), \ (y,s) \in \mathbb{R}^3 \times \mathbb{R}_+.$$

Now we would like to recall the definition of weak solutions of (1.6).

**Definition 1.1.** A pair of functions $(\mathbf{u}, Q) : \Omega \times (0,\infty) \mapsto \mathbb{R}^3 \times \mathcal{S}_0^{(3)}$ is a weak solution of (1.6) and (1.7), if $\mathbf{u} \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(\Omega \times (0,\infty))$ and $Q \in L_t^\infty H_x^1 \cap L_t^2 H_x^2(\Omega \times (0,\infty))$, and for any $\phi \in C_0^\infty(\Omega \times [0,\infty), \mathcal{S}_0^{(3)})$ and $\psi \in C_0^\infty(\Omega \times [0,\infty), \mathbb{R}^3)$, with $\mathrm{div}\psi = 0$ in $\Omega \times [0,\infty)$, it holds that

$$\begin{aligned}
&\int_{\Omega\times(0,\infty)} \left[ -Q \cdot \partial_t\phi - \Delta Q \cdot \phi - Q \cdot \mathbf{u} \otimes \nabla\phi + [Q,\omega]\cdot\phi \right] \mathrm{d}x\mathrm{d}t \\
&\quad = -\int_{\Omega\times(0,\infty)} f_{\mathrm{bulk}}(Q)\cdot\phi \,\mathrm{d}x\mathrm{d}t + \int_\Omega Q_0(x)\cdot\phi(x,0)\,\mathrm{d}x, \qquad (1.10)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Omega\times(0,\infty)} \left[ -\mathbf{u}\cdot\partial_t\psi + \nabla\mathbf{u}\cdot\nabla\psi - \mathbf{u}\otimes\mathbf{u} : \nabla\psi \right] \mathrm{d}x\mathrm{d}t \\
&\quad = \int_{\Omega\times(0,\infty)} \left[ -\Delta Q(\psi\cdot\nabla)Q + [\Delta Q, Q]\cdot\nabla\psi \right] \mathrm{d}x\mathrm{d}t \\
&\qquad + \int_\Omega \mathbf{u}_0(x)\cdot\psi(x,0)\,\mathrm{d}x, \qquad (1.11)
\end{aligned}$$

Paicu–Zarnescu [30] have obtained the existence of global weak solutions to (1.6) and (1.7) in $\mathbb{R}^3$, and the existence of global strong solutions to (1.6) and (1.7) in $\mathbb{R}^2$, when the bulk potential function is $F_{\mathrm{LdG}}(Q)$. Ding-Huang [9] have studied local strong solutions of (1.6). For non-corotational Beris–Edwards system (i.e. $\xi \neq 0$), Paicu–Zarnescu [31] have obtained the existence of global weak solutions to (1.6) and (1.7) in $\mathbb{R}^3$ for sufficiently small $|\xi| > 0$. Later, Cavaterra–Rocca–Wu–Xu [6] have removed the smallness condition on $\xi$ for (1.6) and (1.7) in $\mathbb{R}^2$. Wilkinson [38] has obtained the existence of global weak solutions to (1.6) and (1.7) in three dimensional torus $\mathbb{T}^3$, when the bulk potential function is the Ball–Majumdar potential $F_{\mathrm{BM}}(Q)$. The situation of Beris–Edwards system (1.6) for the De Gennes potential $F_{\mathrm{LdG}}(Q)$ on bounded domains, under the initial-boundary condition, behaves slightly different from that on $\mathbb{R}^3$. In fact, Abels–Dolzmann–Liu [1,2] have established the well-posedness of (1.5) for any arbitrary constant $\xi$; see also [14] for related works on nonisothermal Beris–Edwards system. We also mention an interesting work on the dynamics of $Q$-tensor system by Wu–Xu–Zarnescu [39]. Interested readers can refer to Wang–Zhang–Zhang [37] for a rigorous derivation from Landau–De Gennes theory to Ericksen-Leslie theory. For related works on the existence of global weak solutions to the simplified Ericksen–Leslie system, see [18,26–28].

The works mentioned above left the question open of whether or not certain weak solutions of (1.5) pose either smoothness or partial smoothness properties. This motivates us to study both the existence of suitable weak solutions of (1.6) and their partial regularities. The notion of suitable weak solutions was first introduced by Caffarelli–Kohn–Nirenberg [5] and Scheffer [32] for the Navier–Stokes equation, and later extended by Lin-Liu [24,25] for the simplified Ericksen-Leslie system with variable degree of orientations. Here we introduce the notion of suitable weak solutions to the Beris–Edwards system as follows:

**Definition 1.2.** A weak solution $(\mathbf{u}, P, Q) \in (L_t^\infty L_x^2 \cap L_t^2 H_x^1)(\Omega \times (0,\infty), \mathbb{R}^3) \times L^{\frac{3}{2}}(\Omega \times (0,\infty)) \times (L_t^\infty H_x^1 \cap L_t^2 H_x^2)(\Omega \times (0,\infty), \mathcal{S}_0^{(3)})$ of (1.6) and (1.7) is a suitable weak solution of (1.6), if, in addition, $(\mathbf{u}, P, Q)$ satisfies the local energy inequality $\forall\, 0 \le \phi \in C_0^\infty(\Omega \times (0,t])$,

$$
\begin{aligned}
&\int_\Omega (|\mathbf{u}|^2 + |\nabla Q|^2)\phi(x,t)\,\mathrm{d}x + 2\int_{Q_t} (|\nabla \mathbf{u}|^2 + |\Delta Q|^2)\phi(x,s)\,\mathrm{d}x\mathrm{d}s \\
&\le \int_{Q_t} (|\mathbf{u}|^2 + |\nabla Q|^2)(\partial_t \phi + \Delta\phi)(x,s)\,\mathrm{d}x\mathrm{d}s \\
&\quad + \int_{Q_t} [(|\mathbf{u}|^2 + 2P)\mathbf{u}\cdot\nabla\phi + 2\nabla Q \otimes \nabla Q : \mathbf{u}\otimes\nabla\phi](x,s)\,\mathrm{d}x\mathrm{d}s \\
&\quad +2\int_{Q_t} (\nabla Q \otimes \nabla Q - |\nabla Q|^2 I_3) : \nabla^2\phi(x,s)\,\mathrm{d}x\mathrm{d}s \\
&\quad -2\int_{Q_t} [Q, \Delta Q]\cdot \mathbf{u}\otimes\nabla\phi(x,s)\,\mathrm{d}x\mathrm{d}s \\
&\quad -2\int_{Q_t} [\omega, Q]\cdot(\nabla Q \nabla\phi) + \nabla(f_{\mathrm{bulk}}(Q))\cdot\nabla Q\phi\big](x,s)\,\mathrm{d}x\mathrm{d}s. \qquad (1.12)
\end{aligned}
$$

The notion of suitable weak solutions turns out to be a necessary condition for the smoothness of (1.6). In fact, the local energy inequality (1.12) automatically holds for a sufficiently regular solution of (1.5), which can be obtained by multiplying $(1.5)_2$ by $\mathbf{u}\phi$, and taking spatial derivative of $(1.5)_1$ and multiplying the resulting equation by $\nabla Q\phi$, and then applying integration by parts, see Lemma 2.2 below for the details. We would like to point out that in the process of derivation of (1.12), the cancellation identity

$$\int_{\Omega} [Q, \omega] : \Delta Q \phi \, \mathrm{d}x = - \int_{\Omega} [Q, \Delta Q] : \nabla \mathbf{u} \phi \, \mathrm{d}x \tag{1.13}$$

plays a critical role.

Now we are ready to state our main theorem, which is valid for the Beris–Edwards system associate with both the Landau–De Gennes bulk potential $F_{\mathrm{LdG}}(Q)$ in $\mathbb{R}^3$ and Ball–Majumdar bulk potential $F_{\mathrm{BM}}(Q)$ in $\mathbb{T}^3$. We would like to point out that, due to the technique involving a $L^1 \to L^\infty$ estimate for the advection-diffusion equation on compact manifolds, we choose to work on the domain $\mathbb{T}^3$, instead of $\mathbb{R}^3$, for the Ball–Majumdar potential $F_{\mathrm{BM}}$.

More precisely, we have

**Theorem 1.1.** *For any* $\mathbf{u}_0 \in \mathbf{H}$, *if either*

(i) $\Omega = \mathbb{R}^3$, $F_{\mathrm{bulk}}(\cdot) = F_{\mathrm{LdG}}(\cdot)$ *with* $c > 0$, *and* $Q_0 \in H^1(\mathbb{R}^3, \mathcal{S}_0^{(3)}) \cap L^\infty(\mathbb{R}^3, \mathcal{S}_0^{(3)})$, *or*

(ii) $\Omega = \mathbb{T}^3$, $F_{\mathrm{bulk}}(\cdot) = F_{\mathrm{BM}}(\cdot)$, *and* $Q_0 \in H^1(\mathbb{T}^3, \mathcal{S}_0^{(3)})$ *satisfies* $G_{\mathrm{bulk}}(Q_0) \in L^1(\mathbb{T}^3)$,

*then there exists a global suitable weak solution* $(\mathbf{u}, P, Q) : \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}^3 \times \mathbb{R} \times \mathcal{S}_0^{(3)}$ *of the Beris–Edwards system* (1.6)*, subject to the initial condition* (1.7). *Moreover,*

$$(\mathbf{u}, Q) \in C^\infty(\Omega \times (0, \infty) \setminus \Sigma),$$

*where* $\Sigma \subset \Omega \times \mathbb{R}_+$ *is a closed subset with* $\mathcal{P}^1(\Sigma) = 0$.

We would like to highlight some crucial steps of the proof for Theorem 1.1:

(1) The existence of suitable weak solutions to (1.6) and (1.7) is obtained by modifying the retarded mollification technique, originally due to [32] and [5] in the construction of suitable weak solutions to the Navier–Stokes equation.
(2) For the Landau–De Gennes potential $F_{\mathrm{LdG}}(Q)$, we establish a weak maximum principle of $Q$ for suitable weak solutions $(\mathbf{u}, P, Q)$ of (1.6) and (1.7) that bounds the $L^\infty$-norm of $Q$ in $\mathbb{R}^3 \times (0, \infty)$ in terms of that of initial data $Q_0$; see also [15]. In particular, $\nabla_Q^l f_{\mathrm{LdG}}(Q)$ is also bounded in $\mathbb{R}^3 \times (0, \infty)$ for $l \geq 0$.
(3) For the Ball–Majumdar potential $F_{\mathrm{BM}}(Q)$, we follow the approximation scheme of $G_{\mathrm{BM}}$ by Wilkinson [38] and use the convexity property of $G_{\mathrm{BM}}(Q)$ to bound

$$\|G_{\mathrm{BM}}(Q)\|_{L^\infty(\mathbb{T}^3 \times [\delta, T])}, \ \forall 0 < \delta < T < \infty,$$

in terms of $\|F_{\rm BM}(Q_0)\|_{L^1(\mathbb{T}^3)}$, $\delta$, and $T$. This guarantees that $Q$ is strictly physical in $\mathbb{T}^3 \times [\delta, T]$, i.e., there exists a small $\gamma > 0$, depending on $\delta, T$, such that

$$-\frac{1}{3} + \gamma \leq \lambda_j(Q(x,t)) \leq \frac{2}{3} - \gamma, \ j = 1, 2, 3, \ \forall (x,t) \in \mathbb{T}^3 \times [\delta, T].$$

In particular, both $Q(x,t)$ and $f_{\rm BM}(Q(x,t))$ are bounded in $\mathbb{T}^3 \times [\delta, T]$ for $0 < \delta < T$.

(4) Based on the local energy inequality (1.12), (2), and (3), we perform a blowing up argument to obtain an $\varepsilon_0$-regularity criteria of any suitable weak solution $(\mathbf{u}, P, Q)$ of (1.6), which asserts that if

$$\Phi(z_0, r) := r^{-2} \int_{\mathbb{P}_r(x_0,t_0)} (|\mathbf{u}|^3 + |\nabla Q|^3) \, \mathrm{d}x\mathrm{d}t + \left(r^{-2} \int_{\mathbb{P}_r(x_0,t_0)} |P|^{\frac{3}{2}} \, \mathrm{d}x\mathrm{d}t\right)^2 \leq \varepsilon_0^3, \tag{1.14}$$

then $(x_0, t_0) \in \Omega \times (0, \infty)$ is a smooth point of $(\mathbf{u}, Q)$. The idea is to show that $(\mathbf{u}, P, Q)$ is well approximated by a smooth solution to a linear coupling system in the parabolic neighborhood $\mathbb{P}_{\frac{r}{2}}(x_0, t_0)$ of $(x_0, t_0)$, which heavily relies on the local energy inequality (1.12) and interior $L^{\frac{3}{2}}$-estimate of the pressure function $P$, which turns out to solve the following Poisson equation:

$$-\Delta P = \mathrm{div}^2 \left( \mathbf{u} \otimes \mathbf{u} + \left( \nabla Q \otimes \nabla Q - \frac{1}{2} |\nabla Q|^2 I_3 \right) \right) \text{ in } B_r(x_0). \tag{1.15}$$

Here the following simple identity plays a crucial role in the derivation of (1.15):

$$\mathrm{div}^2 [Q_1, \Delta Q_2 - f_{\rm bulk}(Q_2)] = 0 \text{ in } B_r(x_0), \tag{1.16}$$

for $Q_1, Q_2 \in H^2(B_r(x_0), \mathcal{S}_0^{(3)})$. See §2 for its proof.
This blowing up argument implies that for some $\theta \in (0,1)$, $\Phi_{(x_*,t_*)}(r) \leqq C r^{3\theta}$ for $(x_*, t_*)$ near $(x_0, t_0)$ , which can be used to further show that $(\mathbf{u}, \nabla Q)$ are almost bounded near $(x_0, t_0)$ by an iterated Riesz potential estimates in the parabolic Morrey spaces, see also Huang–Wang [19], Hineman–Wang [17], and Huang–Lin–Wang [18]. Higher order regularity of $(\mathbf{u}, Q)$ near $(x_0, t_0)$ turns out to be more involved than the usual situations, due to the special nonlinearities. Here we establish it by performing higher order energy estimates and utilizing the intrinsic cancellation property, see also [18] for a similar argument on general Ericksen–Leslie system in dimension two. It is well-known [32] that this step is sufficient to show that $(\mathbf{u}, Q)$ is smooth away from a closed set $\Sigma$ which has $\mathcal{P}^{\frac{5}{3}}(\Sigma) = 0$.

(5) To obtain $\mathcal{P}^1(\Sigma) = 0$ from the previous step, we adapt the argument by [5] to show that if

$$\overline{\lim}_{r \to 0} r^{-1} \int_{\mathbb{P}_r(x_0,t_0)} (|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2) \, \mathrm{d}x\mathrm{d}t < \varepsilon_1^2, \tag{1.17}$$

then $(\mathbf{u}, Q) \in C^\infty(\mathbb{P}_{\frac{r}{2}}(x_0, t_0))$. This will be established by extending the so called A, B, C, D Lemmas in [5] to system (1.6).

The paper is organized as follows: in Sect. 2, we derive both the global and local energy inequality for sufficiently regular solutions of (1.6). In Sect. 3, we indicate the construction of suitable weak solutions to (1.6) and (1.7) for both Landau–De Gennes potential and Ball–Majumdar potential. In Sect. 4, we prove two weak maximum principles for suitable weak solutions to (1.6) and (1.7): one for $Q$ and the other for $G_{\mathrm{BM}}(Q)$. In Sect. 5, we prove the first $\varepsilon_0$-regularity of suitable weak solutions to (1.6) and (1.7) in terms of $\Phi(z_0, r)$. In Sect. 6, we will prove the second $\varepsilon_0$-regularity of suitable weak solutions to (1.6) and (1.7) in terms of (1.17).

## 2. Global and Local Energy Inequalities

In this section, we will present proofs for both global energy inequality and local energy inequality for sufficiently regular solutions to the Beris–Edwards system (1.6).

**Lemma 2.1.** *Let* $(\mathbf{u}, Q) \in C^\infty(\Omega \times (0, \infty), \mathbb{R}^3 \times \mathcal{S}_0^{(3)})$ *be a smooth solution of Beris–Edwards system* (1.6)*. Then the global energy inequality* (1.8) *holds.*

*Proof.* The proof is standard, see for instance [30,38]. □

Next we are going to present a local energy inequality for sufficiently regular solutions to the system (1.6).

**Lemma 2.2.** *Assume* $(\mathbf{u}, P, Q) \in C^\infty(\Omega \times (0, \infty), \mathbb{R}^3 \times \mathbb{R} \times \mathcal{S}_0^{(3)})$ *is a smooth solution of* (1.6)*. Then for* $t > 0$ *and any nonnegative* $\phi \in C_0^\infty(\Omega \times (0, t])$*, the following inequality holds on* $Q_t = \Omega \times [0, t]$*:*

$$\begin{aligned}
&\int_\Omega (|\mathbf{u}|^2 + |\nabla Q|^2)\phi(x,t)\,\mathrm{d}x + 2\int_{Q_t}\Big(|\nabla \mathbf{u}|^2 + |\Delta Q|^2\Big)\phi\,\mathrm{d}x\mathrm{d}s \\
&= \int_{Q_t}\Big(|\mathbf{u}|^2 + |\nabla Q|^2\Big)(\partial_t + \Delta)\phi\,\mathrm{d}x\mathrm{d}s \\
&\quad + \int_{Q_t}[(|\mathbf{u}|^2 + 2P)\mathbf{u}\cdot\nabla\phi + 2(\nabla Q\otimes\nabla Q) : \mathbf{u}\otimes\nabla\phi]\,\mathrm{d}x\mathrm{d}s \\
&\quad + 2\int_{Q_t}[(\nabla Q\otimes\nabla Q - |\nabla Q|^2 I_3) : \nabla^2\phi\,\mathrm{d}x\mathrm{d}s \\
&\quad - 2\int_{Q_t}[Q, \Delta Q] : \mathbf{u}\otimes\nabla\phi\,\mathrm{d}x\mathrm{d}s \\
&\quad - 2\int_{Q_t}\Big([\omega, Q] : (\nabla Q\nabla\phi) + \nabla(f_{\mathrm{bulk}}(Q))\cdot\nabla Q\phi\Big)\,\mathrm{d}x\mathrm{d}s.
\end{aligned} \tag{2.1}$$

*Proof.* Using $\operatorname{div}\mathbf{u} = 0$, multiplying the momentum equation (1.6)$_2$ by $\mathbf{u}\phi$, integrating the resulting equation over $\Omega$, and applying integration by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\mathbf{u}|^2\phi\,\mathrm{d}x + \int_\Omega |\nabla\mathbf{u}|^2\phi\,\mathrm{d}x \\
&\quad = \frac{1}{2}\int_\Omega |\mathbf{u}|^2(\partial_t\phi + \Delta\phi)\mathrm{d}x + \frac{1}{2}\int_\Omega (|\mathbf{u}|^2 + 2P)\mathbf{u}\cdot\nabla\phi\,\mathrm{d}x \\
&\qquad - \int_\Omega (\mathbf{u}\cdot\nabla)Q\cdot\Delta Q\phi\,\mathrm{d}x \\
&\qquad - \int_\Omega [Q,\Delta Q] : \nabla\mathbf{u}\phi\,\mathrm{d}x - \int_\Omega [Q,\Delta Q] : \mathbf{u}\otimes\nabla\phi\,\mathrm{d}x.
\end{aligned} \tag{2.2}$$

Taking a spatial derivative of the equation of $Q$ (1.6)$_1$ yields

$$\partial_t\partial_\alpha Q + \mathbf{u}\cdot\nabla\partial_\alpha Q + \partial_\alpha\mathbf{u}\cdot\nabla Q + \partial_\alpha[Q,\omega] = \Delta\partial_\alpha Q - \partial_\alpha(f_{\mathrm{bulk}}(Q)).$$

Using again $\operatorname{div}\mathbf{u} = 0$, multiplying the equation above by $\partial_\alpha Q\phi$, integrating the resulting equation over $\Omega$, and applying integration by parts, and sum over $\alpha$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla Q|^2\phi\,\mathrm{d}x + \int_\Omega |\Delta Q|^2\phi\,\mathrm{d}x \\
&\quad = \frac{1}{2}\int_\Omega |\nabla Q|^2\partial_t\phi\,\mathrm{d}x + \int_\Omega (\mathbf{u}\cdot\nabla)Q\cdot(\Delta Q\phi + \nabla Q\nabla\phi)\,\mathrm{d}x \\
&\qquad - \int_\Omega [\omega, Q] : (\Delta Q\phi + \nabla Q\nabla\phi)\,\mathrm{d}x \\
&\qquad - \int_\Omega \Delta Q\cdot\nabla Q\nabla\phi\,\mathrm{d}x - \int_O \nabla(f_{\mathrm{bulk}}(Q))\cdot\nabla Q\phi\mathrm{d}x.
\end{aligned} \tag{2.3}$$

By direct calculations, there holds

$$\begin{aligned}
&-\int_\Omega \Delta Q\cdot\nabla Q\nabla\phi\,\mathrm{d}x \\
&\quad = \int_\Omega \frac{1}{2}|\nabla Q|^2\Delta\phi\,\mathrm{d}x + \int_\Omega (\nabla Q\otimes\nabla Q - |\nabla Q|^2 I_3) : \nabla^2\phi\,\mathrm{d}x,
\end{aligned} \tag{2.4}$$

and

$$\int_\Omega [\omega, Q] : \Delta Q\phi\,\mathrm{d}x = -\int_\Omega [Q,\Delta Q] : \nabla\mathbf{u}\phi\,\mathrm{d}x. \tag{2.5}$$

Hence, by adding (2.2) and (2.3) together and applying (2.4) and (2.5), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\big(|\mathbf{u}|^2+|\nabla Q|^2\big)\phi\,\mathrm{d}x+\int_{\Omega}\big(|\nabla\mathbf{u}|^2+|\Delta Q|^2\big)\phi\,\mathrm{d}x\\
&\quad=\frac{1}{2}\int_{\Omega}\big(|\mathbf{u}|^2+|\nabla Q|^2\big)(\partial_t+\Delta)\phi\,\mathrm{d}x+\frac{1}{2}\int_{\Omega}(|\mathbf{u}|^2+2P)\mathbf{u}\cdot\nabla\phi\,\mathrm{d}x\\
&\qquad+\int_{\Omega}(\mathbf{u}\cdot\nabla)Q\cdot\nabla Q\nabla\phi\,\mathrm{d}x-\int_{\Omega}[Q,\Delta Q]:\mathbf{u}\otimes\nabla\phi\,\mathrm{d}x\\
&\qquad-\int_{\Omega}[\omega,Q]:\nabla Q\nabla\phi\,\mathrm{d}x-\int_{\Omega}\nabla(f_{\mathrm{bulk}}(Q))\cdot\nabla Q\phi\,\mathrm{d}x\\
&\qquad+\int_{\Omega}(\nabla Q\otimes\nabla Q-|\nabla Q|^2I_3):\nabla^2\phi\,\mathrm{d}x.
\end{aligned}$$

This, after integrating over $[0,t]$, yields the local energy inequality (2.1). □

We close this section by giving a proof of the identity (1.16). More precisely, we have

**Lemma 2.3.** *For $\Omega=\mathbb{R}^3$ or $\mathbb{T}^3$, if $Q^1, Q^2\in H^2(\Omega,\mathcal{S}_0^{(3)})$, then*

$$\mathrm{div}^2[Q^1,\Delta Q^2-f_{\mathrm{bulk}}(Q^2)]=0\ \text{ in }\ \Omega, \tag{2.6}$$

*in the sense of distributions.*

*Proof.* For any $\phi\in C_0^\infty(\Omega)$, we see that

$$\int_{\Omega}\mathrm{div}^2[Q^1,\Delta Q^2-f_{\mathrm{bulk}}(Q^2)](\phi)=\int_{\Omega}[Q^1,\Delta Q^2-f_{\mathrm{bulk}}(Q^2)]_{\alpha\beta})\frac{\partial^2\phi}{\partial x_\alpha\partial x_\beta}\,\mathrm{d}x.$$

Set

$$A_{\alpha\beta}=[Q^1,\Delta Q^2-f_{\mathrm{bulk}}(Q^2)]_{\alpha\beta},\ \ \forall 1\le\alpha,\beta\le 3,$$

and

$$B_{\alpha\beta}=\frac{\partial^2\phi}{\partial x_\alpha\partial x_\beta},\ \ \forall 1\le\alpha,\beta\le 3.$$

Since $Q^1$ and $Q^2$ are symmetric, it is easy to check that

$$A_{\alpha\beta}=-A_{\beta\alpha},\ \ B_{\alpha\beta}=B_{\beta\alpha},\ \ \forall 1\le\alpha,\beta\le 3.$$

We recall the following matrix contraction:

$$A:B=A_{\mathrm{sym}}:B_{\mathrm{sym}}+A_{\mathrm{anti}}:B_{\mathrm{anti}}.$$

Hence (2.6) follows. □

## 3. Global Existence of Suitable Weak Solutions

This section is devoted to the construction of suitable weak solutions to the Beris–Edwards system (1.6). The idea is motived by the "retarded mollification technique" originally due to [32] and [5] in the context of Navier–Stokes equations. Since the procedure for Ball–Majumdar potential $F_{\rm BM}(Q)$ is somewhat different from that for Landau–De Gennes potential $F_{\rm LdG}(Q)$, we will describe them in two separate subsections.

We explain the construction of suitable weak solutions in the spirit of [5]. For $f : \mathbb{R}^4 \to \mathbb{R}$ and $0 < \theta < 1$, define the "retarded mollifier" $\Psi_\theta(f)$ of $f$ by

$$\Psi_\theta[f](x,t) = \frac{1}{\theta^4}\int_{\mathbb{R}^4} \eta\left(\frac{y}{\theta}, \frac{\tau}{\theta}\right) \tilde{f}(x-y, t-\tau)\,\mathrm{d}y\mathrm{d}\tau,$$

where

$$\tilde{f}(x,t) = \begin{cases} f(x,t) & t \geq 0, \\ 0 & t < 0, \end{cases}$$

and the mollifying function $\eta \in C_0^\infty(\mathbb{R}^4)$ satisfies

$$\begin{cases} \eta \geq 0 \quad \text{and} \quad \displaystyle\int_{\mathbb{R}^4} \eta\,\mathrm{d}x\mathrm{d}t = 1, \\ \operatorname{supp}\eta \subset \Big\{(x,t) : |x|^2 < t, \quad 1 < t < 2\Big\}. \end{cases}$$

It follows from Lemma A.8 in [5] that for $\theta \in (0,1]$ and $0 < T \leq \infty$,

$$\begin{aligned} &\mathrm{div}\Psi_\theta[\mathbf{u}] = 0 \quad \text{if} \quad \mathrm{div}\mathbf{u} = 0, \\ &\sup_{0\leq t\leq T}\int_{\mathbb{R}^3} |\Psi_\theta[\mathbf{u}]|^2(x,t)\,\mathrm{d}x \leq C \sup_{0\leq t\leq T}\int_{\mathbb{R}^3} |\mathbf{u}|^2(x,t)\,\mathrm{d}x \\ &\int_{\mathbb{R}^3\times[0,T]} |\nabla\Psi_\theta[\mathbf{u}]|^2(x,t)\,\mathrm{d}x\mathrm{d}t \leq C\int_{\mathbb{R}^3\times[0,T]} |\nabla\mathbf{u}|^2(x,t)\,\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Now we proceed to find the existence of suitable weak solutions of (1.6) and (1.7) as follows:

### *3.1. The Landau–De Gennes potential* $\boxed{F_{\rm bulk}(Q) = F_{\rm LdG}(Q) \text{ and } \Omega = \mathbb{R}^3}$

With the mollifier $\Psi_\theta[\mathbf{u}] \in C^\infty(\mathbb{R}^4)$, we introduce an approximate version of the Beris–Edwards system (1.6), namely,

$$\begin{cases} \partial_t Q^\theta + \mathbf{u}^\theta \cdot \nabla\Psi_\theta[Q^\theta] - [\omega^\theta, \Psi_\theta[Q^\theta]] = \Delta Q^\theta - f_{\rm LdG}(Q^\theta), \\ \partial_t\mathbf{u}^\theta + \Psi_\theta[\mathbf{u}^\theta]\cdot\nabla\mathbf{u}^\theta + \nabla P^\theta \\ = \Delta\mathbf{u}^\theta - \nabla(\Psi_\theta[Q^\theta])\cdot\big(\Delta Q^\theta - f_{\rm LdG}(Q^\theta)\big) \\ \quad +\mathrm{div}[\Psi_\theta[Q^\theta], \Delta Q^\theta - f_{\rm LdG}(Q^\theta)], \\ \mathrm{div}\mathbf{u}^\theta = 0, \end{cases} \quad \text{in } Q_T \quad (3.1)$$

subject to the initial condition (1.7). Here $\omega^\theta = \omega(\mathbf{u}^\theta) = \frac{\nabla \mathbf{u}^\theta - (\nabla \mathbf{u}^\theta)^\top}{2}$.

The idea behind the construction of suitable weak solutions to (3.1) is as follows: for a fixed large $N \geq 1$, set $\theta = \frac{T}{N} \in (0, 1]$, we want to find $\mathbf{u} = \mathbf{u}^\theta$, $P = P^\theta$, and $Q = Q^\theta$ solving (3.1) and (1.7). Since $\Psi_\theta[\mathbf{u}]$ and $\Psi_\theta[Q]$ are smooth, and their values at time $t$ depend only on the values of $\mathbf{u}$ and $Q$ at times prior to $t-\theta$, solving (3.1) and (1.7) involves iteratively solving (3.1) in the interval $[m\theta, (m+1)\theta]$, subject to the initial condition

$$(\mathbf{u}, Q)\big|_{t=m\theta} = (u^\theta, Q^\theta)(\cdot, m\theta) \text{ in } \mathbb{R}^3$$

for $0 \leq m \leq N-1$. This amounts to solving a system that couples a semi-linear parabolic-like equation for $Q$ and a Stokes-like equation for $\mathbf{u}$, in which all the coefficient functions are given smooth functions.

We can verify, by the classical Faedo–Garlekin method, the existence of $(\mathbf{u}^\theta, Q^\theta, P^\theta)$ inductively on each time interval $(m\theta, (m+1)\theta)$ for all $0 \leq m \leq N-1$. Indeed for $m = 0$, according to the definition of $\Psi_\theta$, $\Psi_\theta(\mathbf{u}^\theta) = \Psi_\theta(Q^\theta) = 0$, and the system (3.1) reduces to a linear system

$$\begin{cases} \partial_t Q^\theta = \Delta Q^\theta - f_{\mathrm{LdG}}(Q^\theta) \\ \partial_t \mathbf{u}^\theta + \nabla P^\theta = \Delta \mathbf{u}^\theta \\ \mathrm{div}\mathbf{u}^\theta = 0 \\ (\mathbf{u}^\theta, Q^\theta)|_{t=0} = (\mathbf{u}_0, Q_0) \end{cases} \tag{3.2}$$

in $\mathbb{R}^3 \times [0, \theta]$. For the system (3.2), $Q^\theta$ and $\mathbf{u}^\theta$ are decouple, and $\mathbf{u}^\theta$ can be found according to the standard theory of Stokes equations, while the equation of $Q^\theta$ is a semi-linear parabolic equation which can be solved by the standard method for parabolic equations.

Suppose now that the system (3.1) has been solved for some $0 \leq k < N-1$. We are going to solve the system (3.1):

$$\begin{cases} \partial_t Q_{\alpha\beta} + \mathbf{u} \cdot \nabla \tilde{Q}_{\alpha\beta} - [\omega, \tilde{Q}]_{\alpha\beta} = \Delta Q_{\alpha\beta} - f_{LdG}(Q)_{\alpha\beta} \\ \partial_t \mathbf{u}_\alpha + \tilde{\mathbf{u}} \cdot \nabla \mathbf{u}_\alpha + \partial_\alpha P = \Delta \mathbf{u}_\alpha - \partial_\alpha \tilde{Q}_{\beta\gamma}(\Delta Q - f_{LdG}(Q))_{\beta\gamma} \\ \qquad\qquad + \partial_\beta [\tilde{Q}, \Delta Q - f_{\mathrm{LdG}}(Q)]_{\alpha\beta} \\ \mathrm{div}\mathbf{u} = 0, \end{cases} \tag{3.3}$$

in the time interval $[k\theta, (k+1)\theta]$ with the initial data

$$(\mathbf{u}, Q)|_{t=k\theta} = (\mathbf{u}^\theta, Q^\theta)(\cdot, k\theta) \quad \text{in} \quad \mathbb{R}^3, \tag{3.4}$$

and

$$\tilde{Q} = \Psi_\theta[Q^\theta] \quad \text{and} \quad \tilde{\mathbf{u}} = \Psi_\theta[\mathbf{u}^\theta].$$

Note that $\tilde{\mathbf{u}}$ and $\tilde{Q}$ are smooth functions in $[k\theta, (k+1)\theta] \times \mathbb{R}^3$.

The existence of $(\mathbf{u}, Q)$ in (3.3) may be solved by using the Faedo–Galerkin method. Indeed for a pair of smooth test functions $(\psi, \phi) \in H^2(\mathbb{R}^3, \mathcal{S}_0^{(3)}) \times \mathbf{V}$, the system (3.3) turns to be

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(\nabla Q, \nabla\psi)\,\mathrm{d}x - \int_{\mathbb{R}^3}(\mathbf{u}\cdot\nabla\tilde{Q}, \Delta\psi)\,\mathrm{d}x - \int_{\mathbb{R}^3}([-\omega, \tilde{Q}]_{\alpha\beta}, \Delta\psi_{\alpha\beta})\,\mathrm{d}x\\ &\quad = -\int_{\mathbb{R}^3}(\Delta Q_{\alpha\beta} - f_{LdG}(Q)_{\alpha\beta}, \Delta\psi_{\alpha\beta})\,\mathrm{d}x,\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(\mathbf{u}, \phi)\,\mathrm{d}x + \int_{\mathbb{R}^3}(\tilde{\mathbf{u}}\cdot\nabla\mathbf{u}, \phi)\,\mathrm{d}x + \int_{\mathbb{R}^3}(\nabla\mathbf{u}, \nabla\phi)\,\mathrm{d}x\\ &\quad = -\int_{\mathbb{R}^3}\left(\partial_\alpha\tilde{Q}_{\beta\gamma}(\Delta Q - f_{LdG}(Q))_{\beta\gamma}, \phi_\alpha\right)\mathrm{d}x\\ &\qquad - \int_{\mathbb{R}^3}\left([\tilde{Q}, \Delta Q - f_{\mathrm{LdG}}(Q)]_{\alpha\beta}, \partial_\beta\phi_\alpha\right)\,\mathrm{d}x,\end{aligned} \tag{3.6}$$

in the sense of distributions. The system of first order ODE equations (3.5)–(3.6) can be solved when the test function $(\psi, \phi)$ are taken to be the basis of $H^2(\mathbb{R}^3, \mathcal{S}_0^{(3)}) \times \mathbf{V}$ up to a short time interval $[k\theta, k\theta + T_0]$. Performing the energy estimate for (3.3) as for the original system, we get that, for $k\theta \le t \le k\theta + T_0$,

$$\begin{aligned}&\sup_{t\ge k\theta}\int_{\mathbb{R}^3}\left(|\mathbf{u}^\theta|^2 + |\nabla Q^\theta|^2 + F_{\mathrm{LdG}}(Q^\theta)\right)\,\mathrm{d}x\\ &\quad + \int_{k\theta}^t\int_{\mathbb{R}^3}\left(|\nabla\mathbf{u}^\theta|^2 + |\Delta Q - f_{\mathrm{LdG}}(Q^\theta)|^2\right)\mathrm{d}x\mathrm{d}s\\ &\quad \le \int_{\mathbb{R}^3}\left(|\mathbf{u}^\theta|^2 + |\nabla Q^\theta|^2 + F_{\mathrm{LdG}}(Q^\theta)\right)(x, k\theta)\,\mathrm{d}x.\end{aligned}$$

Hence $T_0$ can be extended up to $\theta$.

Let $(\mathbf{u}^\theta, P^\theta, Q^\theta)$ be the global weak solution of (3.1) and (1.7) in $Q_T$. Then

$$\mathbf{u}^\theta \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_T),\;\; Q^\theta \in L_t^\infty H_x^1 \cap L_t^2 H_x^2(Q_T),\;\; P^\theta \in L^2(Q_T).$$

Observe that

$$[\omega^\theta, \Psi_\theta[Q^\theta]] : (\Delta Q^\theta - f_{\mathrm{LdG}}(Q^\theta)) := -[\Psi_\theta[Q^\theta], \Delta Q^\theta - f_{\mathrm{LdG}}(Q^\theta)] : \nabla\mathbf{u}^\theta.$$

Hence, by calculations similar to Lemma 2.1, we deduce that $(\mathbf{u}^\theta, Q^\theta)$ satisfies the global energy inequality, for $0 \le t \le T$,

$$\begin{aligned}&E(\mathbf{u}^\theta, Q^\theta)(t) + \int_{\mathbb{R}^3\times[0,t]}\left(|\nabla\mathbf{u}^\theta|^2 + |\Delta Q^\theta - f_{\mathrm{LdG}}(Q^\theta)|^2\right)\mathrm{d}x\mathrm{d}t\\ &\quad \le E(\mathbf{u}^\theta, Q^\theta)(0) = \int_{\mathbb{R}^3}\left(\frac{1}{2}|\mathbf{u}_0|^2 + \frac{1}{2}|\nabla Q_0|^2 + F_{\mathrm{LdG}}(Q_0)\right)(x, t)\,\mathrm{d}x.\end{aligned} \tag{3.7}$$

Direct calculations show that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \Delta Q^\theta \cdot f_{\mathrm{LdG}}(Q^\theta)\,\mathrm{d}x \\
&= -a\int_{\mathbb{R}^3} |\nabla Q^\theta|^2\,\mathrm{d}x - c\int_{\mathbb{R}^3} \big(|\nabla Q^\theta|^2|Q^\theta|^2 + \frac{1}{2}|\nabla \mathrm{tr}((Q^\theta)^2)|^2\big)\,\mathrm{d}x \\
&\quad + b\int_{\mathbb{R}^3} \nabla\big((Q^\theta)^2 - \frac{\mathrm{tr}((Q^\theta)^2)}{3}I_3\big)\cdot\nabla Q^\theta\,\mathrm{d}x \\
&\le -\frac{c}{4}\int_{\mathbb{R}^3} \big(|\nabla Q^\theta|^2|Q^\theta|^2 + \frac{1}{2}|\nabla \mathrm{tr}((Q^\theta)^2)|^2\big)\,\mathrm{d}x + C(a,b,c)\int_{\mathbb{R}^3} |\nabla Q^\theta|^2\,\mathrm{d}x.
\end{aligned}$$

This, combined with the assumption $c > 0$ and estimate (3.7), gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} (|\mathbf{u}^\theta|^2 + |\nabla Q^\theta|^2 + F_{\mathrm{LdG}}(Q^\theta))(x,t)\,\mathrm{d}x + 2\int_{\mathbb{R}^3}\Big(|\nabla \mathbf{u}^\theta|^2 + |\Delta Q^\theta|^2\Big)\,\mathrm{d}x \\
&\quad + c\int_{\mathbb{R}^3}\Big(|\nabla Q^\theta|^2|Q^\theta|^2 + \frac{1}{2}|\nabla \mathrm{tr}((Q^\theta)^2)|^2\Big)\,\mathrm{d}x \qquad (3.8)\\
&\le C(a,b,c)\int_{\mathbb{R}^3} |\nabla Q^\theta|^2\,\mathrm{d}x.
\end{aligned}$$

Therefore we deduce from (3.8) and Gronwall's inequality that

$$\begin{aligned}
&\sup_{0\le t\le T}\int_{\mathbb{R}^3} (|\mathbf{u}^\theta|^2 + |\nabla Q^\theta|^2 + F_{\mathrm{LdG}}(Q^\theta))(x,t)\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3\times[0,T]}\Big(|\nabla \mathbf{u}^\theta|^2 + |\Delta Q^\theta|^2\Big)\,\mathrm{d}x\mathrm{d}t \qquad (3.9)\\
&\le C(a,b,c,T)\Big(\|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^3)} + \|Q_0\|^2_{H^1(\mathbb{R}^3)}\Big).
\end{aligned}$$

From (1.1), we know that there exists a $M_0 > 0$, depending on $a, b, c$, such that

$$F_{\mathrm{LdG}}(Q) \ge \frac{c}{2}|Q|^4,\ \forall Q \in \mathcal{S}_0^{(3)} \text{ with } |Q| \ge M_0.$$

This, combined with (3.9) and $F_{\mathrm{LdG}}(Q) \ge 0$, implies that

$$\begin{aligned}
&\sup_{0\le t\le T}\int_{\{x\in\mathbb{R}^3:\ |Q^\theta(x,t)|\ge M_0\}} |Q^\theta(x,t)|^4\,\mathrm{d}x \\
&\le \frac{2}{c}\sup_{0\le t\le T}\int_{\mathbb{R}^3} F_{\mathrm{LdG}}(Q^\theta)(x,t)\,\mathrm{d}x \qquad (3.10)\\
&\le C(a,b,c,T)\big(\|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^3)} + \|Q_0\|^2_{H^1(\mathbb{R}^3)}\big).
\end{aligned}$$

From (3.10), we can conclude that for any compact set $K \subset \mathbb{R}^3$,

$$
\begin{aligned}
&\sup_{0\le t\le T}\int_K |Q^\theta(x,t)|^4\,\mathrm{d}x\\
&\quad\le \sup_{0\le t\le T}\Big\{\int_{\{x\in K:\ |Q^\theta(x,t)|\le M_0\}} |Q^\theta(x,t)|^4\,\mathrm{d}x\\
&\qquad+\int_{\{x\in K:\ |Q^\theta(x,t)|> M_0\}} |Q^\theta(x,t)|^4\,\mathrm{d}x\Big\}\\
&\quad\le |K|M_0^4 + C(a,b,c,T)\big(\|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^3)}+\|Q_0\|^2_{H^1(\mathbb{R}^3)}\big).
\end{aligned}
\tag{3.11}
$$

From (3.9) and (3.11), we have that $\mathbf{u}^\theta$ is uniformly bounded in $L^2_t H^1_x(\mathbb{R}^3\times[0,T])$, $Q^\theta$ is uniformly bounded in $L^2_t H^2_x(K\times[0,T])$ for any compact set $K\subset\mathbb{R}^3$, and $\nabla Q^\theta$ is uniformly bounded in $L^2_t H^1_x(\mathbb{R}^3\times[0,T])$. Therefore, after passing to a subsequence, we may assume that as $\theta\to 0$ (or equivalently $N\to\infty$), there exist $\mathbf{u}\in L^\infty_t L^2_x\cap L^2_t H^1_x(\mathbb{R}^3\times[0,T])$, $Q\in \cap_{R>0}L^\infty_t L^4_x(B_R\times[0,T])$, with $\nabla Q\in L^\infty_t L^2_x\cap L^2_t H^1_x(\mathbb{R}^3\times[0,T])$, such that

$$
\begin{cases}
Q^\theta \rightharpoonup Q & \text{in}\ \ L^2([0,T],L^2(\mathbb{R}^3)),\\
\nabla Q^\theta \rightharpoonup \nabla Q & \text{in}\ \ L^2([0,T],H^1(\mathbb{R}^3)),\\
\mathbf{u}^\theta \rightharpoonup \mathbf{u} & \text{in}\ \ L^2([0,T],H^1(\mathbb{R}^3)).
\end{cases}
\tag{3.12}
$$

Hence by the lower semicontinuity and (3.7) we have that

$$
\begin{aligned}
&E(\mathbf{u},Q)(t)+\int_{\mathbb{R}^3\times[0,t]}\big(|\nabla\mathbf{u}|^2+|\Delta Q-f_{\mathrm{LdG}}(Q)|^2\big)\,\mathrm{d}x\mathrm{d}t\\
&\quad\le E(\mathbf{u},Q)(0)=\int_{\mathbb{R}^3}\left(\frac12|\mathbf{u}_0|^2+\frac12|\nabla Q_0|^2+F_{\mathrm{LdG}}(Q_0)\right)(x,t)\,\mathrm{d}x
\end{aligned}
\tag{3.13}
$$

holds for $0\le t\le T$.

Now we want to estimate the pressure function $P^\theta$. Taking divergence of $(3.1)_2$ gives

$$
\begin{aligned}
-\Delta P^\theta &= \mathrm{div}^2(\Psi_\theta[\mathbf{u}^\theta]\otimes\mathbf{u}^\theta)+\mathrm{div}\big(\nabla(\Psi_\theta[Q^\theta])\cdot(\Delta Q^\theta-f_{\mathrm{LdG}}(Q^\theta))\big)\\
&\quad-\mathrm{div}^2\big([\Psi_\theta[Q^\theta],\Delta Q^\theta-f_{\mathrm{LdG}}(Q^\theta)]\big)\\
&= \mathrm{div}^2(\Psi_\theta[\mathbf{u}^\theta]\otimes\mathbf{u}^\theta)+\mathrm{div}\big(\nabla(\Psi_\theta[Q^\theta])\cdot(\Delta Q^\theta-f_{\mathrm{LdG}}(Q^\theta))\big)\ \text{ in } \mathbb{R}^3.
\end{aligned}
\tag{3.14}
$$

Here we have used in the last step the fact that

$$
\mathrm{div}^2[\Psi_\theta[Q^\theta],\Delta Q^\theta-f_{\mathrm{LdG}}(Q^\theta)]=0\ \text{ in }\ \mathbb{R}^3,
$$

which follows from (1.16).

For $P^\theta$, we claim that $P^\theta$ in $L^{\frac53}(\mathbb{R}^3\times[0,T])$ and

$$
\|P^\theta\|_{L^{\frac53}(\mathbb{R}^3\times[0,T])}\le C\big(a,b,c,T,\|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)},\|Q_0\|_{H^1(\mathbb{R}^3)}\big),\ \forall\theta\in(0,1].
\tag{3.15}
$$

To see this, first observe that (3.9) implies $\nabla(\Psi_\theta[Q^\theta]) \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(\mathbb{R}^3 \times [0, T])$. Hence by the Sobolev interpolation inequality we have that

$$\begin{aligned}\big\|\nabla(\Psi_\theta[Q^\theta])\big\|_{L_t^{10} L_x^{\frac{30}{13}}(\mathbb{R}^3\times[0,T])} &\leqq C\big\|\nabla(\Psi_\theta[Q^\theta])\big\|_{L_t^\infty L_x^2\cap L_t^2 H_x^1(\mathbb{R}^3\times[0,T])}\\ &\leqq C\big(a, b, c, T, \|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)}, \|Q_0\|_{H^1(\mathbb{R}^3)}\big).\end{aligned}$$

By Hölder's inequality, we then have that

$$\begin{aligned}&\big\|\nabla(\Psi_\theta[Q^\theta])\cdot(\Delta Q^\theta - f_{\mathrm{LdG}}(Q^\theta))\big\|_{L_t^{\frac{5}{3}} L_x^{\frac{15}{14}}(\mathbb{R}^3\times[0,T])}\\ &\leq \big\|\nabla(\Psi_\theta[Q^\theta])\big\|_{L_t^{10} L_x^{\frac{30}{13}}(\mathbb{R}^3\times[0,T])}\big\|\Delta Q^\theta - f_{\mathrm{LdG}}(Q^\theta)\big\|_{L^2(\mathbb{R}^3\times[0,T])}\\ &\leqq C\big(a, b, c, T, \|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)}, \|Q_0\|_{H^1(\mathbb{R}^3)}\big).\end{aligned} \tag{3.16}$$

By Calderon-Zygmund's $L^p$-estimate [34], we conclude that $P^\theta \in L^{\frac{5}{3}}([0, T] \times \mathbb{R}^3)$, and

$$\begin{aligned}&\|P^\theta\|_{L^{\frac{5}{3}}([0,T]\times\mathbb{R}^3)}\\ &\leq C\Big[\big\|\Psi_\theta[\mathbf{u}^\theta]\otimes\mathbf{u}^\theta\big\|_{L^{\frac{5}{3}}(\mathbb{R}^3\times[0,T])} + \big\|\nabla(\Psi_\theta[Q^\theta])\cdot(\Delta Q^\theta - f_{\mathrm{LdG}}(Q^\theta))\big\|_{L_t^{\frac{5}{3}} L_x^{\frac{15}{14}}(\mathbb{R}^3\times[0,T])}\Big]\\ &\leq C\Big[\big\|\mathbf{u}^\theta\big\|^2_{L^{\frac{1}{3}}(\mathbb{R}^3\times[0,T])} + \big\|\nabla(\Psi_\theta[Q^\theta])\cdot(\Delta Q^\theta - f_{\mathrm{LdG}}(Q^\theta))\big\|_{L_t^{\frac{5}{3}} L_x^{\frac{15}{14}}(\mathbb{R}^3\times[0,T])}\Big]\\ &\leq C\big(a, b, c, T, \|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)}, \|Q_0\|_{H^1(\mathbb{R}^3)}\big).\end{aligned}$$

It follows from (3.15) that we may assume that there exists $P \in L^{\frac{5}{3}}(\mathbb{R}^3 \times [0, T])$ such that, as $\theta \to 0$,

$$P^\theta \rightharpoonup P \ \text{ in } \ L^{\frac{5}{3}}(\mathbb{R}^3 \times [0, T]). \tag{3.17}$$

From $(3.1)_2$ and the bounds (3.9) and (3.10), we have that

$$\begin{aligned}\partial_t\mathbf{u}^\theta &= -\Psi_\theta[\mathbf{u}^\theta]\cdot\nabla\mathbf{u}^\theta - \nabla P^\theta + \Delta\mathbf{u}^\theta - \nabla(\Psi_\theta[Q^\theta])\cdot(\Delta Q^\theta - f_{\mathrm{LdG}}(Q^\theta))\\ &\quad+\mathrm{div}([\Psi_\theta[Q^\theta], \Delta Q^\theta - f_{\mathrm{LdG}}(Q^\theta)])\\ &\in L^{\frac{5}{4}}(\mathbb{R}^3\times[0,T]) + L^{\frac{5}{3}}([0,T], W^{-1,\frac{5}{3}}(\mathbb{R}^3)) + \bigcap_{R>0} L^2([0,T], W^{-1,\frac{4}{3}}(B_R)),\end{aligned}$$

and for any $0 < R < \infty$,

$$\begin{aligned}&\Big\|\partial_t\mathbf{u}^\theta\Big\|_{L^{\frac{5}{4}}(\mathbb{R}^3\times[0,T])+L^{\frac{5}{3}}([0,T],W^{-1,\frac{5}{3}}(\mathbb{R}^3))+L^2([0,T],W^{-1,\frac{4}{3}}(B_R))}\\ &\quad\leqq C\big(a, b, c, R, T, \|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)}, \|Q_0\|_{H^1(\mathbb{R}^3)}\big),\ \forall\theta\in(0,1].\end{aligned} \tag{3.18}$$

Similarly, it follows from $(3.1)_1$ and the bounds (3.9) and (3.10) that $\partial_t Q^\theta \in L^{\frac{5}{3}}(\mathbb{R}^3 \times [0, T]) + \bigcap_{R>0} L^2([0, T], L^{\frac{4}{3}}(B_R))$, and

$$\begin{aligned}&\Big\|\partial_t Q^\theta\Big\|_{L^{\frac{5}{3}}(\mathbb{R}^3\times[0,T])+L^2([0,T],L^{\frac{4}{3}}(B_R))}\\ &\quad\leq C\big(a, b, c, R, T, \|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)}, \|Q_0\|_{H^1(\mathbb{R}^3)}\big)\end{aligned} \tag{3.19}$$

for all $0 < R < \infty$ and $\theta \in (0, 1]$.

By (3.9), (3.10), (3.18) and (3.19), we can apply Aubin–Lions' compactness Lemma ([35]) to conclude that, for any $0 < R < \infty$,

$$\big(\mathbf{u}^\theta, Q^\theta, \nabla Q^\theta\big) \to \big(\mathbf{u}, Q, \nabla Q\big) \quad \text{in} \quad L^3(B_R \times [0, T]), \quad \text{as } \theta \to 0. \tag{3.20}$$

On the other hand, it follows from $F_{\mathrm{LdG}}(Q^\theta) \geq 0$ in $\mathbb{R}^3 \times [0, T]$ and (3.9) that

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} |\nabla Q^\theta|^2(x, t)\, \mathrm{d}x \leq C\big(a, b, c, T, \|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)}, \|Q_0\|_{H^1(\mathbb{R}^3)}\big).$$

Hence, by (3.20), we also have that for any $1 < p_1 < 6$ and $1 < p_2 < \frac{10}{3}$,

$$Q^\theta \to Q \text{ in } L^{p_1}(B_R \times [0, T]); \; \mathbf{u}^\theta \to \mathbf{u} \; \text{ in } L^{p_2}(B_R \times [0, T]) \; \text{ as } \; \theta \to 0. \tag{3.21}$$

With the convergences (3.12), (3.17), and (3.20), it is not hard to show that the limit $(\mathbf{u}, P, Q)$ is a weak solution of (1.6) and (1.7), i.e., it satisfies the system (1.6) and (1.7) in the sense of distributions (see also [30] Proposition 3). We leave the details to interested readers, apart from pointing out that in the sense of distributions, as $\theta \to 0$,

$$\nabla P^\theta - \nabla(\Psi_\theta[Q^\theta]) \cdot f_{\mathrm{LdG}}(Q^\theta) \to \nabla P - \nabla Q \cdot f_{\mathrm{LdG}}(Q) = \nabla(P - F_{\mathrm{LdG}}(Q)).$$

To show that $(\mathbf{u}, P, Q)$ is a suitable weak solution of (1.6), observe that, as in Lemma 2.2, we can test equations of $\mathbf{u}^\theta$ in (3.1) by $\mathbf{u}^\theta\phi$, and taking a spatial derivative of the equation of $Q^\theta$ in (3.1) and then testing it by $\nabla Q^\theta \phi$ for any nonnegative $\phi \in C_0^\infty(\mathbb{R}^3 \times (0, t])$, to obtain the following local energy inequality:

$$\begin{aligned}
&\int_{\mathbb{R}^3} \big(|\mathbf{u}^\theta|^2 + |\nabla Q^\theta|^2\big)\phi(x, t)\, \mathrm{d}x + 2\int_0^t \int_{\mathbb{R}^3} \big(|\nabla \mathbf{u}^\theta|^2 + |\Delta Q^\theta|^2\big)\phi \, \mathrm{d}x\mathrm{d}s \\
&\quad = \int_0^t \int_{\mathbb{R}^3} \Big[\big(|\mathbf{u}^\theta|^2 + |\nabla Q^\theta|^2\big)(\partial_t \phi + \Delta\phi) + 2\nabla \Psi_\theta[Q^\theta] \otimes \nabla Q^\theta : \mathbf{u}^\theta \otimes \nabla\phi\Big] \mathrm{d}x\mathrm{d}s \\
&\qquad + \int_0^t \int_{\mathbb{R}^3} (|\mathbf{u}^\theta|^2 \Psi_\theta[\mathbf{u}^\theta] \cdot \nabla\phi + 2P^\theta \mathbf{u}^\theta \cdot \nabla\phi + 2\nabla(\Psi_\theta[Q^\theta]) \cdot f_{\mathrm{LdG}}(Q^\theta)\mathbf{u}^\theta \phi)\, \mathrm{d}x\mathrm{d}s \\
&\qquad + 2\int_0^t \int_{\mathbb{R}^3} \big([\Psi_\theta[Q^\theta], f_{\mathrm{LdG}}(Q^\theta)]\big) : \nabla \mathbf{u}^\theta \phi \, \mathrm{d}x\mathrm{d}s \\
&\qquad + 2\int_0^t \int_{\mathbb{R}^3} \big(\nabla Q^\theta \otimes \nabla Q^\theta - |\nabla Q^\theta|^2 I_3)\big) : \nabla^2\phi \, \mathrm{d}x\mathrm{d}s \\
&\qquad - 2\int_0^t \int_{\mathbb{R}^3} ([\Psi_\theta[Q^\theta], \Delta Q^\theta - f_{\mathrm{LdG}}(Q^\theta)]) : \mathbf{u}^\theta \otimes \nabla\phi \, \mathrm{d}x\mathrm{d}s \\
&\qquad - 2\int_0^t \int_{\mathbb{R}^3} [\omega^\theta, \Psi_\theta[Q^\theta]] : \nabla Q^\theta \nabla\phi \, \mathrm{d}x\mathrm{d}s \\
&\qquad - 2\int_0^t \int_{\mathbb{R}^3} \nabla(f_{\mathrm{LdG}}(Q^\theta)) \cdot \nabla Q^\theta \phi \, \mathrm{d}x\mathrm{d}s.
\end{aligned} \tag{3.22}$$

Taking the limit in (3.22) as $\theta \to 0$, we see by the lower semicontinuity that it holds that

$$\int_{\mathbb{R}^3} \left(|\mathbf{u}|^2 + |\nabla Q|^2\right)\phi(x,t)\,\mathrm{d}x + 2\int_0^t \int_{\mathbb{R}^3} \left(|\nabla \mathbf{u}|^2 + |\Delta Q|^2\right)\phi\,\mathrm{d}x\mathrm{d}s$$
$$\leq \liminf_{\theta\to 0}\Big[\int_{\mathbb{R}^3} \left(|\mathbf{u}^\theta|^2 + |\nabla Q^\theta|^2\right)\phi(x,t)\,\mathrm{d}x$$
$$+ 2\int_0^t \int_{\mathbb{R}^3} \left(|\nabla \mathbf{u}^\theta|^2 + |\Delta Q^\theta|^2\right)\phi\,\mathrm{d}x\mathrm{d}s\Big],$$

while it follows from (3.20) and (3.21) that

$$\lim_{\theta\to 0} \text{ Right hand side of } (3.22)$$
$$= \int_0^t \int_{\mathbb{R}^3} \Big(|\mathbf{u}|^2 + |\nabla Q|^2\Big)(\partial_t\phi + \Delta\phi)\,\mathrm{d}x\mathrm{d}t$$
$$+ \int_0^t \int_{\mathbb{R}^3} (|\mathbf{u}|^2 + |\nabla Q|^2 + 2(P - F_{\mathrm{LdG}}(Q)))\mathbf{u}\cdot\nabla\phi)$$
$$+ 2\nabla Q\otimes\nabla Q : \mathbf{u}\otimes\nabla\phi\,\mathrm{d}x\mathrm{d}s$$
$$+ 2\int_0^t \int_{\mathbb{R}^3} \left[\nabla Q\otimes\nabla Q - |\nabla Q|^2 I_3\right] : \nabla^2\phi\,\mathrm{d}x\mathrm{d}s$$
$$- 2\int_0^t \int_{\mathbb{R}^3} [Q, \Delta Q] : \mathbf{u}\otimes\nabla\phi\,\mathrm{d}x\mathrm{d}s$$
$$- 2\int_0^t \int_{\mathbb{R}^3} \left(\omega Q - Q\omega\right) : \nabla Q\nabla\phi\,\mathrm{d}x\mathrm{d}s - 2\int_0^t \int_{\mathbb{R}^3} \nabla(f_{\mathrm{LdG}}(Q))\cdot\nabla Q\phi\,\mathrm{d}x\mathrm{d}s.$$

Here we have used the following convergence result:

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^3} \nabla(\Psi_\theta[Q^\theta])\cdot f_{\mathrm{LdG}}(Q^\theta)\mathbf{u}^\theta\phi\,\mathrm{d}x\mathrm{d}s &\to \int_0^t \int_{\mathbb{R}^3} \nabla Q\cdot f_{\mathrm{LdG}}(Q)\mathbf{u}\phi\,\mathrm{d}x\mathrm{d}s\\ &= \int_0^t \int_{\mathbb{R}^3} \nabla(F_{\mathrm{LdG}}(Q))\mathbf{u}\phi\,\mathrm{d}x\mathrm{d}s\\ &= -\int_0^t \int_{\mathbb{R}^3} F_{\mathrm{LdG}}(Q)\mathbf{u}\nabla\phi\,\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{3.23}$$

Putting these together yields the desired local energy inequality (1.12) for $(\mathbf{u}, P, Q)$. This completes the proof of the existence of suitable weak solution in the first case. □

In the next subsection, we will indicate how to construct a suitable weak solution of (3.1) for the Ball–Majumdar potential function.

### *3.2. The Ball–Majumdar potential* $F_{\mathrm{bulk}}(Q) = F_{\mathrm{BM}}(Q)$ *and* $\Omega = \mathbb{T}^3$

Since $G_{\mathrm{BM}}$, given by (1.3), is singular outside the physical domain

$$\mathcal{D} = \left\{Q \in \mathcal{S}_0^{(3)} : -\frac{1}{3} < \lambda_i(Q) < \frac{2}{3},\ i = 1, 2, 3\right\},$$

we need to regularize it. For this part, we follow the scheme by Wilkinson [38] (Sect. 3) very closely. First we regularize it by using the Yosida–Moreau regularization of convex analysis [11,36]: For $m \in \mathbb{N}^+$, define

$$\widetilde{G}^m_{\mathrm{BM}}(Q) := \inf_{A \in \mathcal{S}_0^{(3)}} \Big\{ m|A - Q|^2 + G_{\mathrm{BM}}(A) \Big\}, \ \forall Q \in \mathcal{S}_0^{(3)}.$$

Then smoothly mollify $\widetilde{G}^m_{\mathrm{BM}}$ through the standard mollifications:

$$G^m_{\mathrm{BM}}(Q) := \int_{\mathcal{S}_0^{(3)}} \widetilde{G}^m_{\mathrm{BM}}(Q - R)\Phi_m(R)\, dR,$$

where $\Phi_m(R) = m^5 \Phi\,(mR)$, and $\Phi \in C_0^\infty(\mathcal{S}_0^{(3)})$ is nonnegative and satisfies

$$\mathrm{supp}\ \Phi \subset \Big\{ Q \in \mathcal{S}_0^{(3)} : \ |Q| < 1 \Big\}, \ \int_{\mathcal{S}_0^{(3)}} \Phi(R)\, dR = 1.$$

As in [38] Proposition 3.1, $G^m_{\mathrm{BM}}$ satisfies the following properties:

(G0) *$G^m_{\mathrm{BM}}$ is an isotropic function of $Q$.*
(G1) *$G^m_{\mathrm{BM}} \in C^\infty(\mathcal{S}_0^{(3)})$ is convex on $\mathcal{S}_0^{(3)}$.*
(G2) *There exists a constant $g_0 > 0$, independent of $m$, such that for any $m \in \mathbb{N}^+$, $G^m_{\mathrm{BM}}(Q) \geq -g_0$ holds for all $Q \in \mathcal{S}_0^{(3)}$.*
(G3) *$G^m_{\mathrm{BM}}(Q) \leq G^{m+1}_{\mathrm{BM}}(Q) \leq G_{\mathrm{BM}}(Q)$ on $\mathcal{S}_0^{(3)}$ for all $m \geq 1$.*
(G4) *$G^m_{\mathrm{BM}} \to G_{\mathrm{BM}}$ and $\nabla_Q G^m_{\mathrm{BM}} \to \nabla_Q G_{\mathrm{BM}}$ in $L^\infty_{\mathrm{loc}}(\mathcal{D})$, as $m \to \infty$.*
(G5) *There exist $\alpha(m), \beta(m), \gamma(m) > 0$ such that*

$$\alpha(m)|Q| - \beta(m) \leq \Big|\langle \nabla_Q G^m_{\mathrm{BM}}(Q)\rangle\Big| \leq \gamma(m)(1 + |Q|), \ \forall Q \in \mathcal{S}_0^{(3)}.$$

(G6) *For $k \geq 2$,there exists $C(m,k) > 0$ such that*

$$\Big|\langle \nabla^k_Q G^m_{\mathrm{BM}}(Q)\rangle\Big| \leq C(m,k)(1 + |Q|^2), \ \forall Q \in \mathcal{S}_0^{(3)}.$$

For our purpose in this paper, we also need the following estimate on $G^m_{\mathrm{BM}}$.

**Lemma 3.1.** *For any $m \in \mathbb{N}^+$, $G^m_{\mathrm{BM}}$ satisfies*

$$G^m_{\mathrm{BM}}(Q) \geq \frac{m}{4}|Q|^2 - g_0, \ \forall Q \in \mathcal{S}_0^{(3)} \text{ with } |Q| \geq 11, \tag{3.24}$$

*where $g_0 > 0$ is the same constant given by* (G2).

*Proof.* Since $G_{\mathrm{BM}}(Q) = \infty$ for $Q \notin \mathcal{D}$, it follows from the definition of $\widetilde{G}^m_{\mathrm{BM}}$ and (G2) that

$$\begin{aligned}\widetilde{G}^m_{\mathrm{BM}}(Q) &= \inf_{A \in \mathcal{D}} \Big\{ m|A - Q|^2 + G_{\mathrm{BM}}(A) \Big\} \\ &\geq \inf_{A \in \mathcal{D}} \Big\{ m|A - Q|^2 \Big\} - g_0 \\ &= m\mathrm{dist}^2\big(Q, \overline{\mathcal{D}}\big) - g_0.\end{aligned}$$

Thus for any $Q \in \mathcal{S}_0^{(3)}$ with $|Q| \geq 10$, we have

$$\widetilde{G}_{\mathrm{BM}}^m(Q) \geq m(|Q| - \frac{2}{\sqrt{3}})^2 - g_0 \geq m\big(\frac{|Q|}{\sqrt{2}}\big)^2 - g_0 = \frac{m}{2}|Q|^2 - g_0.$$

It is not hard to see that this estimate, along with the definition of $G_{\mathrm{BM}}^m$, yields (3.24). The proof is now complete. □

Now we set

$$F_{\mathrm{BM}}^m(Q) = G_{\mathrm{BM}}^m(Q) - \frac{\kappa}{2}|Q|^2, \ \forall Q \in \mathcal{S}_0^{(3)},$$

and

$$f_{\mathrm{BM}}^m(Q) = \big\langle \nabla_Q G_{\mathrm{BM}}^m(Q) \big\rangle - \kappa Q, \ \forall Q \in \mathcal{S}_0^{(3)}.$$

Observe that the convexity of $G_{\mathrm{BM}}^m$ on $\mathcal{S}_0^{(3)}$ yields that

$$\mathrm{tr} \nabla_Q f_{\mathrm{BM}}^m(Q)(\nabla Q, \nabla Q) = \mathrm{tr} \nabla_Q^2 F_{\mathrm{BM}}^m(Q)(\nabla Q, \nabla Q) \geq -\kappa |\nabla Q|^2, \quad (3.25)$$

for all $Q \in H^1(\Omega, \mathcal{S}_0^{(3)})$.

Note that if we view a function on $\mathbb{T}^3$ as a $\mathbb{Z}^3$- periodic function on $\mathbb{R}^3$, then the “retarded” mollification procedure given in the previous subsection can be directly performed on functions defined in $\mathbb{T}^3$.

Similar to the Sect. 3.1, we can introduce an approximate system of (3.1) for the Ball–Majumdar potential as follows. For $T > 0$ and a fixed large $N \in \mathbb{N}^+$, let $\theta = \frac{T}{N} \in (0, 1]$. Then we seek $(\mathbf{u}^{\theta,m}, P^{\theta,m}, Q^{\theta,m})$ that solves

$$\begin{cases} \partial_t Q^{\theta,m} + \mathbf{u}^{\theta,m} \cdot \nabla \Psi_\theta[Q^{\theta,m}] - [\omega^{\theta,m}, \Psi_\theta[Q^{\theta,m}]] \\ = \Delta Q^{\theta,m} - f_{\mathrm{BM}}^m(Q^{\theta,m}), \\ \partial_t \mathbf{u}^{\theta,m} + \Psi_\theta[\mathbf{u}^{\theta,m}] \cdot \nabla \mathbf{u}^{\theta,m} + \nabla P^{\theta,m} \\ = \Delta \mathbf{u}^{\theta,m} - \nabla(\Psi_\theta[Q^{\theta,m}]) \cdot \big(\Delta Q^{\theta,m} - f_{\mathrm{BM}}^m(Q^{\theta,m})\big) \\ + \mathrm{div}\, \big([\Psi_\theta[Q^{\theta,m}], \Delta Q^{\theta,m} - f_{\mathrm{BM}}^m(Q^{\theta,m})]\big), \\ \mathrm{div} \mathbf{u}^{\theta,m} = 0, \end{cases} \quad (3.26)$$

in $\mathbb{T}^3 \times [0, T]$, subject to the initial condition (1.7). Here $\omega^{\theta,m} = \omega(\mathbf{u}^{\theta,m}) = \frac{\nabla \mathbf{u}^{\theta,m} - (\nabla \mathbf{u}^{\theta,m})^\top}{2}$.

Since the system (3.26) is simply the system (3.1) with $f_{\mathrm{LdG}}$ replaced by $f_{\mathrm{BM}}^m$, we can argue as in the Sect. 3.1 to find a global weak solution $(\mathbf{u}^{\theta,m}, P^{\theta,m}, Q^{\theta,m})$ of (3.26) and (1.7) in $Q_T = \mathbb{T}^3 \times [0, T]$ such that

$$\mathbf{u}^{\theta,m} \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_T), \ \ Q^{\theta,m} \in L_t^\infty H_x^1 \cap L_t^2 H_x^2(Q_T), \ \ P^{\theta,m} \in L^2(Q_T).$$

Moreover, by calculations similar to Lemma 2.1, we deduce that $(\mathbf{u}^{\theta,m}, Q^{\theta,m})$ satisfies the global energy inequality, for $0 \leq t \leq T$,

$$\begin{aligned} E(\mathbf{u}^{\theta,m}, Q^{\theta,m})(t) &+ \int_{\mathbb{T}^3 \times [0,t]} \Big(|\nabla \mathbf{u}^{\theta,m}|^2 + |\Delta Q^{\theta,m} - f_{\mathrm{BM}}^m(Q^{\theta,m})|^2\Big)\, \mathrm{d}x\mathrm{d}t \\ &= E(\mathbf{u}^{\theta,m}, Q^{\theta,m})(0) \leq \int_{\mathbb{T}^3} \Big(\frac{1}{2}|\mathbf{u}_0|^2 + \frac{1}{2}|\nabla Q_0|^2 + F_{\mathrm{BM}}(Q_0)\Big)(x)\, \mathrm{d}x. \end{aligned} \quad (3.27)$$

It follows from (3.27) and (3.25) that

$$\begin{aligned}
&\int_{\mathbb{T}^3\times[0,t]} |\Delta Q^{\theta,m} - f^m_{\mathrm{BM}}(Q^{\theta,m})|^2 \,\mathrm{d}x\mathrm{d}t \\
&= \int_{\mathbb{T}^3\times[0,t]} \Big(|\Delta Q^{\theta,m}|^2 + |f^m_{\mathrm{BM}}(Q^{\theta,m})|^2 - 2\Delta Q^{\theta,m}\cdot f^m_{\mathrm{BM}}(Q^{\theta,m})\Big)\,\mathrm{d}x\mathrm{d}t \\
&= \int_{\mathbb{T}^3\times[0,t]} \Big(|\Delta Q^{\theta,m}|^2 + |f^m_{\mathrm{BM}}(Q^{\theta,m})|^2 + 2\mathrm{tr}\nabla_Q f^m_{\mathrm{BM}}(Q^{\theta,m})(\nabla Q^{\theta,m}, \nabla Q^{\theta,m})\Big)\,\mathrm{d}x\mathrm{d}t \\
&\geq \int_{\mathbb{T}^3\times[0,t]} \Big(|\Delta Q^{\theta,m}|^2 + |f^m_{\mathrm{BM}}(Q^{\theta,m})|^2 - \kappa|\nabla Q^{\theta,m}|^2\Big)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Substituting this into (3.27) and applying Gronwall's inequality, we obtain that for any $0 \leq t \leq T$,

$$\begin{aligned}
&E(\mathbf{u}^{\theta,m}, Q^{\theta,m})(t) + \int_{\mathbb{T}^3\times[0,t]} \big(|\nabla \mathbf{u}^{\theta,m}|^2 + |\Delta Q^{\theta,m}|^2 + |f^m_{\mathrm{BM}}(Q^{\theta,m})|^2\big)\,\mathrm{d}x\mathrm{d}t \\
&\quad \leq e^{CT}\int_{\mathbb{T}^3} \Big(\frac{1}{2}|\mathbf{u}_0|^2 + \frac{1}{2}|\nabla Q_0|^2 + F_{\mathrm{BM}}(Q_0)\Big)(x)\,\mathrm{d}x.
\end{aligned} \tag{3.28}$$

It follows from (3.27) that

$$\sup_{0\leq t\leq T}\int_{\mathbb{T}^3} F^m_{\mathrm{BM}}(Q^{\theta,m})(x,t)\,\mathrm{d}x \leq \int_{\mathbb{T}^3}\Big(\frac{1}{2}|\mathbf{u}_0|^2 + \frac{1}{2}|\nabla Q_0|^2 + F_{\mathrm{BM}}(Q_0)\Big)(x)\,\mathrm{d}x.$$

This, combined with (G2) and (3.24), implies that there exists a sufficiently large $m_0 = m_0(\kappa, g_0) \in \mathbb{N}^+$ such that, for all $m \geq m_0$,

$$\begin{aligned}
&(\frac{m}{8} - \frac{\kappa}{2})\int_{\{x\in\mathbb{T}^3:\ |Q^{\theta,m}(x,t)|\geq 11\}} |Q^{\theta,m}|^2(x,t)\,\mathrm{d}x \\
&\quad\leq \int_{\{x\in\mathbb{T}^3:\ |Q^{\theta,m}(x,t)|\geq 11\}} \Big[\Big(\frac{m}{4}|Q^{\theta,m}|^2 - g_0\Big) - \frac{\kappa}{2}|Q^{\theta,m}|^2\Big](x,t)\,\mathrm{d}x \\
&\quad\leq \int_{\{x\in\mathbb{T}^3:\ |Q^{\theta,m}(x,t)|\geq 11\}} F^m_{\mathrm{BM}}(Q^{\theta,m})(x,t)\,\mathrm{d}x \\
&\quad= \int_{\mathbb{T}^3} F^m_{\mathrm{BM}}(Q^{\theta,m})(x,t)\,\mathrm{d}x - \int_{\{x\in\mathbb{T}^3:\ |Q^{\theta,m}(x,t)|\leq 11\}} F^m_{\mathrm{BM}}(Q^{\theta,m})(x,t)\,\mathrm{d}x \\
&\quad= \int_{\mathbb{T}^3} F^m_{\mathrm{BM}}(Q^{\theta,m})(x,t)\,\mathrm{d}x \\
&\qquad - \int_{\{x\in\mathbb{T}^3:\ |Q^{\theta,m}(x,t)|\leq 11\}} \Big[\Big(G^m_{\mathrm{BM}}(Q^{\theta,m}) + g_0\Big) - \frac{\kappa}{2}|Q^{\theta,m}|^2 - g_0\Big](x,t)\,\mathrm{d}x \\
&\quad\leq \int_{\mathbb{T}^3} F^m_{\mathrm{BM}}(Q^{\theta,m})(x,t)\,\mathrm{d}x + \int_{\{x\in\mathbb{T}^3:\ |Q^{\theta,m}(x,t)|\leq 11\}} (g_0 + \frac{\kappa}{2}|Q^{\theta,m}|^2(x,t))\,\mathrm{d}x \\
&\quad\leq \int_{\mathbb{T}^3} (\frac{1}{2}|\mathbf{u}_0|^2 + \frac{1}{2}|\nabla Q_0|^2 + F_{\mathrm{BM}}(Q_0)\big)(x)\,\mathrm{d}x + (g_0 + \frac{121\kappa}{2})|\mathbb{T}^3|
\end{aligned}$$

holds for any $0 \leq t \leq T$. Therefore we conclude that for $m \geq m_0$, it holds that

$$\begin{aligned}
&\sup_{0\leq t\leq T}\int_{\mathbb{T}^3} |Q^{\theta,m}|^2(x,t)\,\mathrm{d}x \\
&\quad\leq C\Big(\|\mathbf{u}_0\|_{L^2(\mathbb{T}^3)}, \|Q_0\|_{H^1(\mathbb{T}^3)}, \|F_{\mathrm{BM}}(Q_0)\|_{L^1(\mathbb{T}^3)},, g_0, \kappa\Big).
\end{aligned} \tag{3.29}$$

As in Sect. 3.1, the pressure function $P^{\theta,m}$ solves

$$\begin{aligned} &-\Delta P^{\theta,m} \\ &= \operatorname{div}^2\big(\Psi_\theta[\mathbf{u}^{\theta,m}]\otimes\mathbf{u}^{\theta,m}\big)+\operatorname{div}\big(\nabla(\Psi_\theta[Q^{\theta,m}])\cdot \\ &(\Delta Q^{\theta,m}-f^m_{\mathrm{BM}}(Q^{\theta,m}))\big) \end{aligned} \quad \text{in } \mathbb{T}^3. \tag{3.30}$$

We can apply the same argument as in the previous subsection to conclude that $P^{\theta,m}\in L^{\frac{5}{3}}(\mathbb{T}^3\times[0,T])$, and

$$\|P^{\theta,m}\|_{L^{\frac{5}{3}}(\mathbb{T}^3\times[0,T])}\le C\Big(\|\mathbf{u}_0\|_{L^2(\mathbb{T}^3)},\|Q_0\|_{H^1(\mathbb{T}^3)},\|F_{\mathrm{BM}}(Q_0)\|_{L^1(\mathbb{T}^3)},,g_0,\kappa\Big). \tag{3.31}$$

With estimates (3.31) and (3.28), we can utilize the system (3.26) to obtain that

$$\begin{aligned} &\Big\|\partial_t\mathbf{u}^{\theta,m}\Big\|_{L^2([0,T],W^{-1,4}(\mathbb{T}^3))} \\ &\leqq C\Big(\|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)},\|Q_0\|_{H^1(\mathbb{R}^3)},\|F_{\mathrm{BM}}(Q_0)\|_{L^1(\mathbb{T}^3)},,g_0,\kappa\Big), \end{aligned} \tag{3.32}$$

$$\begin{aligned} &\Big\|\partial_t Q^{\theta,m}\Big\|_{L^2([0,T],L^{\frac{3}{2}}(\mathbb{T}^3))} \\ &\le C\Big(\|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)},\|Q_0\|_{H^1(\mathbb{R}^3)},\|F_{\mathrm{BM}}(Q_0)\|_{L^1(\mathbb{T}^3)},,g_0,\kappa\Big), \end{aligned} \tag{3.33}$$

uniformly for $\theta\in(0,1]$ and $m\ge m_0$.

For each fixed $m\ge m_0$, we can assume without loss of generality that there exists

$$(\mathbf{u}^m,P^m,Q^m)\in L^\infty_tL^2_x\cap L^2_tH^1_x(Q_T)\times L^{\frac{5}{3}}(Q_T)\times L^\infty_tH^1_x(Q_T)$$

such that as $\theta\to0$,

$$\begin{cases} \mathbf{u}^{\theta,m}\rightharpoonup\mathbf{u}^m \ \text{ in } \ L^2_tH^1_x(Q_T),\\ \mathbf{u}^{\theta,m}\to\mathbf{u}^m \ \text{ in } \ L^p(Q_T)\ \forall 1<p<\frac{10}{3},\\ P^{\theta,m}\rightharpoonup P^m \ \text{ in } \ L^{\frac{5}{3}}(Q_T),\\ Q^{\theta,m}\rightharpoonup Q^m \ \text{ in } \ L^2_tH^2_x(Q_T),\\ Q^{\theta,m}\to Q^m \ \text{ in } \ L^r_tL^s_x(Q_T),\ \forall 1<r,s<\infty,\\ \Delta Q^{\theta,m}-f^m_{\mathrm{BM}}(Q^{\theta,m})\rightharpoonup\Delta Q^m-f^m_{\mathrm{BM}}(Q^m) \ \text{ in } \ L^2(Q_T),\\ F^m_{\mathrm{BM}}(Q^{\theta,m})\to F^m_{\mathrm{BM}}(Q^m) \ \text{ in } \ L^1(Q_T). \end{cases}$$

As in Sect. 3.1, we can now verify that $(\mathbf{u}^m,P^m,Q^m)$ is a weak solution of

$$\begin{cases} \partial_tQ^m+\mathbf{u}^m\cdot\nabla Q^m-[\omega^m,Q^m]=\Delta Q^m-f^m_{\mathrm{BM}}(Q^m),\\ \partial_t\mathbf{u}^m+\mathbf{u}^m\cdot\nabla\mathbf{u}^m+\nabla(P^m-F^m_{\mathrm{BM}}(Q))\\ =\Delta\mathbf{u}^m-\nabla Q^m\cdot\Delta Q^m+\operatorname{div}[Q^m,\Delta Q^m],\\ \operatorname{div}\mathbf{u}^m=0, \end{cases} \tag{3.34}$$

in $\mathbb{T}^3\times[0,T]$, subject to the initial condition (1.7).

By the lower semicontinuity the following global energy inequality holds: for $0 \le t \le T$,

$$\int_{\mathbb{T}^3} (\frac{1}{2}|\mathbf{u}^m|^2 + \frac{1}{2}|\nabla Q^m|^2 + F^m_{\rm BM}(Q^m))(x,t)\,\mathrm{d}x + \int_{\mathbb{T}^3\times[0,t]} \big(|\nabla \mathbf{u}^m|^2 + |\Delta Q^m - f^m_{\rm BM}(Q^m)|^2\big)\,\mathrm{d}x\mathrm{d}t \le \int_{\mathbb{T}^3} (\frac{1}{2}|\mathbf{u}_0|^2 + \frac{1}{2}|\nabla Q_0|^2 + F_{\rm BM}(Q_0))(x)\,\mathrm{d}x, \tag{3.35}$$

and

$$E(\mathbf{u}^m, Q^m)(t) + \int_{\mathbb{T}^3\times[0,t]} \big(|\nabla \mathbf{u}^m|^2 + |\Delta Q^m|^2 + |f^m_{\rm BM}(Q^m)|^2\big)\,\mathrm{d}x\mathrm{d}t \le e^{CT}\int_{\mathbb{T}^3} (\frac{1}{2}|\mathbf{u}_0|^2 + \frac{1}{2}|\nabla Q_0|^2 + F_{\rm BM}(Q_0))(x)\,\mathrm{d}x,\ \forall t\in[0,T]. \tag{3.36}$$

Also it follows from (3.29), (3.31), (3.32) and (3.36) that

$$\max\Big\{\|Q^m\|_{L^\infty_t L^2(Q_T)}, \|P^m\|_{L^{\frac{5}{3}}(Q_T)}, \|\partial_t \mathbf{u}^m\|_{L^2_t W^{-1,4}_x(Q_T)}, \|\partial_t Q^m\|_{L^2_t L^{\frac{3}{2}}_x(Q_T)}\Big\} \le C\Big(\|\mathbf{u}_0\|_{L^2(\mathbb{T}^3)}, \|Q_0\|_{H^1(\mathbb{T}^3)}, \|F_{\rm BM}(Q_0)\|_{L^1(\mathbb{T}^3)},, g_0,\kappa\Big). \tag{3.37}$$

Furthermore, we can check that $(\mathbf{u}^m, P^m, Q^m)$ is a suitable weak solution of (3.34) by verifying that it satisfies the local inequality (1.12) with $f_{\rm bulk}$ replaced by $f^m_{\rm BM}$.

To show that as $m\to\infty$, $(\mathbf{u}^m, P^m, Q^m)$ gives rise to a suitable weak solution of (3.1), we need to first show that $Q^m$ lies in a strictly physical subdomain of the physical domain $\mathcal{D}$, since $G_{\rm BM}(Q)$ blows up as $Q\in\mathcal{D}$ tends to $\partial\mathcal{D}$. This amounts to establishing an $L^\infty$-estimate of $G_{\rm BM}(Q)$ in terms of the $L^1$-norm of $G_{\rm BM}(Q_0)$, which was previously shown by Wilkinson [38] in a slightly different setting.

More precisely, we need the following version of a generalized maximum principle:

**Lemma 3.2.** *There exist $m_0\in\mathbb{N}^+$ and a positive constant $C_0$, independent of $m$, such that for all $m\ge m_0$,*

$$\big\|G^m_{\rm BM}(Q^m)(\cdot,t)\big\|_{L^\infty(\mathbb{T}^3)} \le C_0 t^{-\frac{5}{2}}\big\|G_{\rm BM}(Q_0)\big\|_{L^1(\mathbb{T}^3)} + C_0,\ \forall 0<t<T. \tag{3.38}$$

For now we assume Lemma 3.2, which will be proved in §4 below. We may assume without loss of generality that there exists

$$(\mathbf{u}, P, Q) \in L^\infty_t L^2_x \cap L^2_t H^1_x(Q_T) \times L^{\frac{5}{3}}(Q_T) \times L^\infty_t H^1_x \cap L^2_t H^2_x(Q_T)$$

such that

$$\begin{cases} \mathbf{u}^m \rightharpoonup \mathbf{u} & \text{in } L^2_t H^1_x(Q_T),\\ \mathbf{u}^m \to \mathbf{u} & \text{in } L^p(Q_T),\ \forall 1<p<\frac{10}{3},\\ P^m \rightharpoonup P & \text{in } L^{\frac{5}{3}}(Q_T),\\ Q^m \rightharpoonup Q & \text{in } L^2_t H^2_x(Q_T),\\ Q^m \to Q & \text{in } L^r_t L^s_x(Q_T),\ \forall 1<r,s<\infty. \end{cases}$$

From (3.38), we can also deduce that for any $0 < \delta < T$,

$$\|G_{\mathrm{BM}}(Q)\|_{L^\infty(\mathbb{T}^3\times[\delta,T])} \le (C\delta^{-\frac{5}{2}} + e^T)\|G_{\mathrm{BM}}(Q_0)\|_{L^1(\mathbb{T}^3)} + \kappa^2 e^T. \quad (3.39)$$

By the logarithmic divergence of $G_{\mathrm{BM}}$ as $Q \in \mathcal{D} \to \partial\mathcal{D}$ and (3.39), we conclude that for any $\delta > 0$, there exists $\varepsilon_0 = \varepsilon_0(\delta, T) > 0$ such that

$$Q(x,t) \in \mathcal{D}_{\varepsilon_0},\ \forall (x,t) \in \mathbb{T}^3 \times [\delta, T], \quad (3.40)$$

where

$$\mathcal{D}_{\varepsilon_0} := \left\{ Q \in \mathcal{D} :\ -\frac{1}{3} + \varepsilon_0 \le \lambda_i(Q(x,t)) \le \frac{2}{3} - \varepsilon_0,\ i = 1,2,3 \right\}. \quad (3.41)$$

From (3.38) and the quadratic growth property of $G^m_{\mathrm{BM}}$, we also see that there exists $C_0 > 0$, independent of $m$, such that for $m \ge m_0$,

$$|Q^m(x,t)| \le C_0,\ (x,t) \in \mathbb{T}^3 \times [\delta, T]. \quad (3.42)$$

We now claim that

$$f^m_{\mathrm{BM}}(Q^m) \rightharpoonup f_{\mathrm{BM}}(Q) \text{ in } L^2(\mathbb{T}^3 \times [\delta, T]),\ \text{as } m \to \infty. \quad (3.43)$$

To see this, first observe that (3.36) yields that $f^m_{\mathrm{BM}}(Q^m)$ is uniformly bounded in $L^2(\mathbb{T}^3 \times [0,T])$. Thus there exists a function $\bar f \in L^2(\mathbb{T}^3 \times [0,T])$ such that

$$f^m_{\mathrm{BM}}(Q^m) \rightharpoonup \bar f \in L^2(\mathbb{T}^3 \times [0,T]).$$

Now we want to identify $\bar f$. It follows from $Q^m \to Q$ in $L^2(\mathbb{T}^3 \times [0,T])$ that there exists $E_m \subset \mathbb{T}^3 \times [0,T]$, with $|E_m| \to 0$, such that

$$Q^m \to Q,\ \text{uniformly in } \mathbb{T}^3 \times [0,T] \setminus E_m,$$

which, combined with $Q(\mathbb{T}^3 \times [\delta, T]) \subset \mathcal{D}_{\varepsilon_0}$, yields that for sufficiently large $m$,

$$Q^m(\mathbb{T}^3 \times [\delta, T] \setminus E_m) \subset \mathcal{D}_{\frac{\varepsilon_0}{2}}.$$

Since $f^m_{\mathrm{BM}} \to f_{\mathrm{BM}}$ in $W^{1,\infty}(\mathcal{D}_{\frac{\varepsilon_0}{2}})$, we conclude that

$$f^m_{\mathrm{BM}}(Q^m) \to f_{\mathrm{BM}}(Q),\ \text{uniformly in } \mathbb{T}^3 \times [\delta, T] \setminus E_m.$$

Therefore $\bar f = f_{\mathrm{BM}}(Q)$ for a.e. $(x,t) \in \mathbb{T}^3 \times [0,T]$, and (3.43) holds.

From (3.43) and $\Delta Q^m \rightharpoonup \Delta Q$ in $L^2(\mathbb{T}^3 \times [0,T])$, as $m \to \infty$, we see that

$$\Delta Q^m - f^m_{\mathrm{BM}}(Q^m) \rightharpoonup \Delta Q - f_{\mathrm{BM}}(Q) \text{ in } L^2(\mathbb{T}^3 \times [0,T]),\ \text{as } m \to \infty,$$

With all the estimates in hand, it is rather standard to show that passing to the limit in (3.34), as $m \to \infty$ first and $\delta \to 0$ second, yields that $(\mathbf{u}, P, Q)$ is a weak solution of (3.1). While passing to the limit in the local inequality for $(\mathbf{u}^m, P^m, Q^m)$, as $m \to \infty$ first and then $\delta \to 0$, we can also verify that $(\mathbf{u}, P, Q)$ satisfies the local energy inequality (1.12) with $f_{\mathrm{bulk}}(Q)$ replaced by $f_{\mathrm{BM}}(Q)$. □

## 4. Maximum Principles

In this section, we will show the maximum principles for any weak solution $(\mathbf{u}, Q)$ of (1.6) and (1.7) in $\mathbb{R}^3$ with the Landau-De Gennes potential function $F_{\text{LdG}}(Q)$ (see also [15,16]), and in $\mathbb{T}^3$ with the Ball–Majumdar potential function $F_{\text{BM}}(Q)$ (see also [38]). These will play important roles in the proof of partial regularity of suitable weak solutions to (1.6) in the Sects. 5 and 6 below.

**Lemma 4.1.** *For* $(\mathbf{u}_0, Q_0) \in \mathbf{H}\times H^1(\mathbb{R}^3, \mathcal{S}_0^{(3)})$, *let* $(\mathbf{u}, Q) \in L_t^2 H_x^1(\mathbb{R}^3\times\mathbb{R}_+, \mathbb{R}^3)\times L_t^2 H_x^2(\mathbb{R}^3 \times \mathbb{R}_+, \mathcal{S}_0^{(3)})$ *be a weak solution of* (1.6)–(1.7). *If, in addition,* $Q_0 \in L^\infty(\mathbb{R}^3, \mathbb{S}_0^{(3)})$ *and* $c > 0$, *then there exists a constant* $C > 0$, *depending on* $\|Q_0\|_{L^\infty(\mathbb{R}^3)}$ *and* $a, b, c$, *such that*

$$|Q(x,t)| \le C,\ \forall (x,t) \in \mathbb{R}^3 \times \mathbb{R}_+. \tag{4.1}$$

*Proof.* This is a well-known fact. The readers can find the proof in [15,16] or [30]. □

Next we will give a proof of Lemma 3.2, which guarantees that $Q$ lies inside a strictly physical subdomain $\mathcal{D}_{\varepsilon_0}$ so that $F_{\text{BM}}(Q)$ becomes regular and hence $f_{\text{BM}}(Q)$ is bounded.

*Proof of Lemma 3.2.* It follows from the chain rule and the equation $(3.34)_1$ that $G^m_{\text{BM}}(Q^m)$ satisfies, in the weak sense, that

$$\begin{aligned}
&\partial_t(G^m_{\text{BM}}(Q^m)) + \mathbf{u}^m\cdot\nabla(G^m_{\text{BM}}(Q^m))\\
&= \Delta(G^m_{\text{BM}}(Q^m)) - \text{tr}\nabla^2_Q G^m_{\text{BM}}(Q^m)(\nabla Q^m, \nabla Q^m) - f^m_{\text{BM}}(Q^m)\langle\nabla_Q G^m_{\text{BM}}(Q^m)\rangle,\\
&\le \Delta(G^m_{\text{BM}}(Q^m)) - (\langle\nabla_Q G^m_{\text{BM}}(Q^m) - \kappa Q^m)\langle\nabla_Q G^m_{\text{BM}}(Q^m)\\
&\le \Delta(G^m_{\text{BM}}(Q^m)) + \frac{\kappa^2}{2}|Q^m|^2
\end{aligned} \tag{4.2}$$

in $\mathbb{T}^3\times(0,T]$. Indeed, this can be obtained by multiplying $(3.34)_1$ by $\langle\nabla_Q G^m_{\text{BM}}(Q^m)\rangle$ and using the fact $G^m_{\text{BM}}$ is a smooth convex function. Therefore $G^m_{\text{BM}}(Q^m) \in L_t^\infty H_x^1(\mathbb{T}^3 \times [0,T])$ satisfies, in the weak sense, that

$$\begin{aligned}
&\partial_t(G^m_{\text{BM}}(Q^m)) + \mathbf{u}^m\cdot\nabla(G^m_{\text{BM}}(Q^m))\\
&\quad\le \Delta(G^m_{\text{BM}}(Q^m)) + \frac{\kappa^2}{2}|Q^m|^2,
\end{aligned} \qquad \text{in } \mathbb{T}^3\times(0,T]. \tag{4.3}$$

It follows from (3.35) and (3.37) that $Q^m \in L_t^2 H_x^2(\mathbb{T}^3 \times [0,T])$. In particular, by Sobolev's embedding theorem, we have that

$$\begin{aligned}
&\|Q^m\|_{L_t^2 L_x^\infty(\mathbb{T}^3\times[0,T])}\\
&\le C\Big(\|\mathbf{u}_0\|_{L^2(\mathbb{T}^3)}, \|Q_0\|_{H^1(\mathbb{T}^3)}, \|F_{\text{BM}}(Q_0)\|_{L^1(\mathbb{T}^3)}, , g_0, \kappa\Big).
\end{aligned} \tag{4.4}$$

Since the drifting coefficient $\mathbf{u}^m$ in (4.3) is not smooth and $Q^m$ is not bounded in $\mathbb{T}^3 \times [0,T]$, we can not directly apply the argument of §8 in [38] to prove 3.38.

Here we proceed it by first considering an auxiliary equation with mollifying $\mathbf{u}^m$ as the drifting coefficient. More precisely, let $\mathbf{u}_\epsilon^m$ be a standard $\epsilon$-mollification on $\mathbb{T}^3 \times [0, T]$ for $0 < \epsilon < 1$. Then $\mathbf{u}_\epsilon^m \in C^\infty(\mathbb{T}^3 \times [0, T])$ satisfies $\text{div}\mathbf{u}_\epsilon^m = 0$ and

$$\mathbf{u}_\epsilon^m \to \mathbf{u}^m \text{ in } L_t^2 H_x^1(\mathbb{T}^3 \times [0, T]), \text{ as } \epsilon \to 0.$$

Also let $g_\epsilon^m$ be $\epsilon$-mollifications of $|Q^m|^2$ in $\mathbb{T}^3 \times [0, T]$, and $h_\epsilon^m$ be $\epsilon$-mollifications of $G_{\text{BM}}^m(Q_0)$ in $\mathbb{T}^3$. Then it follows from (4.4) that for all $m \geq m_0$,

$$\|g^m\|_{L_t^2 L_x^\infty(\mathbb{T}^3\times[0,T])} \leq \|Q^m\|^2_{L_t^2 L_x^\infty(\mathbb{T}^3\times[0,T])},$$
$$\|h_\epsilon^m\|_{L^1(\mathbb{T}^3)} \leq \|G_{\text{BM}}(Q_0)\|_{L^1(\mathbb{T}^3)},$$

and

$$g_\epsilon^m \to |Q^m|^2 \text{ in } L^3(\mathbb{T}^3 \times [0, T]),\ h_\epsilon^m \to G_{\text{BM}}^m(Q_0) \text{ in } L^1(\mathbb{T}^3), \text{ as } \epsilon \to 0.$$

Now let $v_\epsilon^m \in C^\infty(\mathbb{T}^3 \times [0, T])$ be the unique solution of

$$\begin{cases} \partial_t v_\epsilon^m + \mathbf{u}_\epsilon^m \cdot \nabla v_\epsilon^m = \Delta v_\epsilon^m + \dfrac{\kappa^2}{2} g_\epsilon^m & \text{in } \mathbb{T}^3 \times [0, T], \\ v_\epsilon^m = h_\epsilon^m & \text{on } \mathbb{T}^3 \times \{0\}. \end{cases} \tag{4.5}$$

For $v_\epsilon^m$, we will modify the argument as illustrated in [38], §8, to achieve that for $0 < t < T$,

$$\|v_\epsilon^m(\cdot, t)\|_{L^\infty(\mathbb{T}^3)} \leq Ct^{-\frac{5}{2}} \|G_{\text{BM}}(Q_0)\|_{L^1(\mathbb{T}^3)} + C_0. \tag{4.6}$$

To show (4.6), decompose $v_\epsilon^m = v_1 + v_2$, where $v_1$ solves

$$\begin{cases} \partial_t v_1 + \mathbf{u}_\epsilon^m \cdot \nabla v_1 = \Delta v_1, & \text{in } \mathbb{T}^3 \times [0, T], \\ v_1 = h_\epsilon^m - \displaystyle\int_{\mathbb{T}^3} h_\epsilon^m, & \text{on } \mathbb{T}^3 \times \{0\}, \end{cases} \tag{4.7}$$

and $v_2$ solves

$$\begin{cases} \partial_t v_2 + \mathbf{u}_\epsilon^m \cdot \nabla v_2 = \Delta v_2 + \dfrac{\kappa^2}{2} g_\epsilon^m, & \text{in } \mathbb{T}^3 \times [0, T], \\ v_2 = \displaystyle\int_{\mathbb{T}^3} h_\epsilon^m, & \text{on } \mathbb{T}^3 \times \{0\}. \end{cases} \tag{4.8}$$

For $v_1$, we can apply the $L^1 \to L^\infty$ estimate for advection-diffusion equations on compact manifold [7] as in Lemma 8.1 of [38] to conclude that

$$\|v_1(\cdot, t)\|_{L^\infty(\mathbb{T}^3)} \leq Ct^{-\frac{5}{2}} \Big\|h_\epsilon^m - \int_{\mathbb{T}^3} h_\epsilon^m\Big\|_{L^1(\mathbb{T}^3)} \leq Ct^{-\frac{5}{2}} \|G_{\text{BM}}(Q_0)\|_{L^1(\mathbb{T}^3)} \tag{4.9}$$

for $0 < t < T$.

While for $v_2$, we can multiply $(4.8)_1$ by $|v_2|^{p-2}v_2$, $p > 2$, and integrate the resulting equation over $\mathbb{T}^3$ to get

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\left\|v_2(t)\right\|_{L^p(\mathbb{T}^3)}^p \leq \frac{\kappa^2}{2}\left\|g_\epsilon^m(t)\right\|_{L^p(\mathbb{T}^3)}\left\|v_2(t)\right\|_{L^p(\mathbb{T}^3)}^{p-1}$$
$$\leq \frac{\kappa^2}{2}\left|\mathbb{T}^3\right|^{\frac{1}{p}}\left\|g_\epsilon^m(t)\right\|_{L^\infty(\mathbb{T}^3)}\left\|v_2(t),\right.$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|v_2(t)\right\|_{L^p(\mathbb{T}^3)} \leq \frac{\kappa^2}{2}\left|\mathbb{T}^3\right|^{\frac{1}{p}}\left\|g_\epsilon^m(t)\right\|_{L^\infty(\mathbb{T}^3)},$$

and hence

$$\left\|v_2(t)\right\|_{L^p(\mathbb{T}^3)} \leq \left\|v_2(0)\right\|_{L^p(\mathbb{T}^3)} + \frac{\kappa^2}{2}\left|\mathbb{T}^3\right|^{\frac{1}{p}}\int_0^T \left\|g_\epsilon^m(t)\right\|_{L^\infty(\mathbb{T}^3)}\,\mathrm{d}t,\ \forall 0 < t \leq T.$$

Sending $p \to \infty$ and applying (4.4), we obtain that for $0 < t < T$,

$$\begin{aligned}&\left\|v_2(t)\right\|_{L^\infty(\mathbb{T}^3)}\\ &\leq C\|h_\epsilon^m\|_{L^1(\mathbb{T}^3)} + \frac{\kappa^2}{2}\int_0^T \|Q^m(t)\|_{L^\infty(\mathbb{T}^3)}^2\,\mathrm{d}t \\ &\leq \left\|G_{\mathrm{BM}}(Q_0)\right\|_{L^1(\mathbb{T}^3)} + C\Big(\|\mathbf{u}_0\|_{L^2(\mathbb{T}^3)}, \|Q_0\|_{H^1(\mathbb{T}^3)}, \|F_{\mathrm{BM}}(Q_0)\|_{L^1(\mathbb{T}^3)}, , g_0, \kappa\Big).\end{aligned} \tag{4.10}$$

Putting (4.9) and (4.10) together yields (4.6).

It is not hard to see that as $\epsilon \to 0$, there exists $v^m \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(\mathbb{T}^3 \times [0, T])$ such that $v_\epsilon^m \to v^m$ in $L^2(\mathbb{T}^3 \times [0, T])$. Passing to the limit in the equation (4.5), we see that $v^m$ is a weak solution of

$$\begin{cases}\partial_t v^m + \mathbf{u}^m \cdot \nabla v^m = \Delta v^m + \dfrac{\kappa^2}{2}|Q^m|^2 & \text{in } \mathbb{T}^3 \times [0, T],\\ v^m = G_{\mathrm{BM}}^m(Q_0) & \text{on } \mathbb{T}^3 \times \{0\}.\end{cases} \tag{4.11}$$

Moreover, passing to the limit of (4.6), we have that for any $0 < t < T$,

$$\left\|v^m(\cdot, t)\right\|_{L^\infty(\mathbb{T}^3)} \leq Ct^{-\frac{5}{2}}\left\|G_{\mathrm{BM}}(Q_0)\right\|_{L^1(\mathbb{T}^3)} + C_0. \tag{4.12}$$

Now observe that by the comparison principle on (4.3), we know that for $m \geq m_0$, it holds.

$$G_{\mathrm{BM}}^m(Q^m)(x, t) \leq v^m(\cdot, t) \leq Ct^{-\frac{5}{2}}\left\|G_{\mathrm{BM}}(Q_0)\right\|_{L^1(\mathbb{T}^3)} + C_0,$$

for all $(x, t) \in \mathbb{T}^3 \times [0, T]$. This, combined with (G2), yields (3.38). □

Note that passing to the limit in (3.38), the suitable weak solution $(\mathbf{u}, P, Q)$ to (3.1), constructed in §3.2, satisfies that for any $0 < \delta < T$,

$$\left\|G_{\mathrm{BM}}(Q)\right\|_{L^\infty(\mathbb{T}^3\times[\delta,T])} \leq C_0\delta^{-\frac{5}{2}}\left\|G_{\mathrm{BM}}(Q_0)\right\|_{L^1(\mathbb{T}^3)} + C_0. \tag{4.13}$$

This completes the proof of Lemma 3.2. □

## 5. Partial Regularity; Part I

This section is devoted to establishing an $\epsilon_0$-regularity for suitable weak solutions $(\mathbf{u}, Q)$ of (1.6) in $\Omega \times (0, \infty)$ in terms of renormalized $L^3$-norm of $(\mathbf{u}, Q)$. The argument we will present is based on a blowing up argument, motivated by that of Lin [23] on the Navier–Stokes equation, which works equally well for both the Landau–De Gennes potential $F_{\text{LdG}}$ and the Ball–Majumdar potential $F_{\text{BM}}$. More precisely, we want to establish the following property:

**Lemma 5.1.** *For any $M > 0$, there exist $\varepsilon_0 > 0$, $0 < \tau_0 < \frac{1}{2}$, and $C_0 > 0$, depending on $M$, such that if $(\mathbf{u}, Q, P)$ is a suitable weak solution of (1.6) in $\Omega \times (0, \infty)$, which satisfies, for $z_0 = (x_0, t_0) \in \Omega \times (r^2, \infty)$ and $r > 0$,*

$$\begin{cases} |Q| \leq M & \text{if } F_{\text{bulk}} = F_{\text{LdG}} \text{ and } \Omega = \mathbb{R}^3, \\ |G_{\text{BM}}(Q)| \leq M & \text{if } F_{\text{bulk}} = F_{\text{BM}} \text{ and } \Omega = \mathbb{T}^3, \end{cases} \quad \text{in } \mathbb{P}_r(z_0), \tag{5.1}$$

*and*

$$r^{-2} \int_{\mathbb{P}_r(z_0)} (|\mathbf{u}|^3 + |\nabla Q|^3) \, \mathrm{d}x\mathrm{d}t + \Big( r^{-2} \int_{\mathbb{P}_r(z_0)} |P|^{\frac{3}{2}} \, \mathrm{d}x\mathrm{d}t \Big)^2 \leq \varepsilon_0^3, \tag{5.2}$$

*then*

$$\begin{aligned} &(\tau_0 r)^{-2} \int_{\mathbb{P}_{\tau_0 r}(z_0)} (|\mathbf{u}|^3 + |\nabla Q|^3) \, \mathrm{d}x\mathrm{d}t + \Big( (\tau_0 r)^{-2} \int_{\mathbb{P}_{\tau_0 r}(z_0)} |P|^{\frac{3}{2}} \, \mathrm{d}x\mathrm{d}t \Big)^2 \\ &\leq \frac{1}{2} \max \Big\{ r^{-2} \int_{\mathbb{P}_r(z_0)} (|\mathbf{u}|^3 + |\nabla Q|^3) \, \mathrm{d}x\mathrm{d}t \\ &\quad + \Big( r^{-2} \int_{\mathbb{P}_r(z_0)} |P|^{\frac{3}{2}} \, \mathrm{d}x\mathrm{d}t \Big)^2, C_0 r^3 \Big\}. \end{aligned} \tag{5.3}$$

*Proof.* We prove it by contradiction. Suppose that the conclusion were false. Then there exists $M_0 > 0$ such that for any $\tau \in (0, \frac{1}{2})$, we can find $\varepsilon_i \to 0$, $C_i \to \infty$, and $r_i > 0$, and $z_i = (x_i, t_i) \in \mathbb{R}^3 \times (r_i^2, \infty)$ such that

$$\begin{cases} |Q| \leq M_0 & \text{if } F_{\text{bulk}} = F_{\text{LdG}}, \\ |G_{\text{BM}}(Q)| \leq M_0 & \text{if } F_{\text{bulk}} = F_{\text{BM}}, \end{cases} \quad \text{in } \mathbb{P}_{r_i}(z_i), \tag{5.4}$$

and

$$r_i^{-2} \int_{\mathbb{P}_{r_i}(z_i)} (|\mathbf{u}|^3 + |\nabla Q|^3) \, \mathrm{d}x\mathrm{d}t + \big( r_i^{-2} \int_{\mathbb{P}_{r_i}(z_i)} |P|^{\frac{3}{2}} \, \mathrm{d}x\mathrm{d}t \big)^2 = \varepsilon_i^3, \tag{5.5}$$

but

$$\begin{aligned} &(\tau r_i)^{-2} \int_{\mathbb{P}_{\tau r_i}(z_i)} (|\mathbf{u}|^3 + |\nabla Q|^3) \, \mathrm{d}x\mathrm{d}t + \big( (\tau r_i)^{-2} \int_{\mathbb{P}_{\tau r_i}(z_i)} |P|^{\frac{3}{2}} \, \mathrm{d}x\mathrm{d}t \big)^2 \\ &> \frac{1}{2} \max \big\{ \varepsilon_i^3, C_i r_i^3 \big\}. \end{aligned} \tag{5.6}$$

From (5.6), we see that

$$\begin{aligned} C_i r_i^3 &\le 2(\tau r_i)^{-2} \int_{\mathbb{P}_{\tau r_i}(z_i)} (|\mathbf{u}|^3 + |\nabla Q|^3)\,\mathrm{d}x\mathrm{d}t + 2\Big((\tau r_i)^{-2} \int_{\mathbb{P}_{\tau r_i}(z_i)} |P|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t\Big)^2 \\ &\le 2\tau^{-4} \left\{ r_i^{-2} \int_{\mathbb{P}_{r_i}(z_i)} (|\mathbf{u}|^3 + |\nabla Q|^3)\,\mathrm{d}x\mathrm{d}t + \Big(r_i^{-2} \int_{\mathbb{P}_{r_i}(z_i)} |P|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t\Big)^2 \right\} \\ &= 2\tau^{-4}\varepsilon_i^3, \end{aligned}$$

so that

$$r_i \le \Big(\frac{2\varepsilon_i^3}{C_i\tau^4}\Big)^{\frac{1}{3}} \to 0.$$

Also from (5.4), we know that there exist $C_0 > 0$ and $\delta_0 > 0$ such that, in the case $F_{\text{bulk}} = F_{\text{BM}}$,

$$Q(z) \in \mathcal{D}_{\delta_0} \text{ and } |f_{\text{BM}}(Q(z))| + |\nabla_Q f_{\text{BM}}(Q(z))| \le C_0,\ \forall z \in \mathbb{P}_{r_i}(z_i). \tag{5.7}$$

Define a rescaled sequence of maps

$$(\mathbf{u}_i, Q_i, P_i)(x,t) = \big(r_i\mathbf{u}, Q, r_i^2 P\big)(x_i + r_i x, t_i + r_i^2 t),\ \forall x \in \mathbb{R}^3,\ t > -1.$$

Then $(\mathbf{u}_i, Q_i, P_i)$ is a weak solution of the scaled Beris–Edwards system

$$\begin{cases} \partial_t Q_i + \mathbf{u}_i \cdot \nabla Q_i - [\omega(\mathbf{u}_i), Q_i] = \Delta Q_i - r_i^2 f_{\text{bulk}}(Q_i), \\ \partial_t \mathbf{u}_i + \mathbf{u}_i \cdot \nabla \mathbf{u}_i + \nabla P_i = \Delta \mathbf{u}_i - \nabla Q_i \cdot \Delta Q_i - \operatorname{div}[\Delta Q_i, Q_i], \\ \operatorname{div}\mathbf{u}_i = 0, \end{cases} \tag{5.8}$$

where

$$\omega(\mathbf{u}_i) = \frac{\nabla \mathbf{u}_i - (\nabla \mathbf{u}_i)^T}{2}.$$

Moreover, $(\mathbf{u}_i, Q_i, P_i)$ satisfies

$$\int_{\mathbb{P}_1(0)} (|\mathbf{u}_i|^3 + |\nabla Q_i|^3)\,\mathrm{d}x\mathrm{d}t + \left(\int_{\mathbb{P}_1(0)} |P_i|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t\right)^2 = \varepsilon_i^3, \tag{5.9}$$

and

$$\begin{aligned} &\tau^{-2} \int_{\mathbb{P}_\tau(0)} (|\mathbf{u}_i|^3 + |\nabla Q_i|^3)\,\mathrm{d}x\mathrm{d}t + \left(\tau^{-2} \int_{\mathbb{P}_\tau(0)} |P_i|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t\right)^2 \\ &\quad > \frac{1}{2} \max\left\{\varepsilon_i^3, C_i r_i^3\right\}. \end{aligned} \tag{5.10}$$

Define the blowing-up sequence $(\widehat{\mathbf{u}}_i, \widehat{Q}_i, \widehat{P}_i) : \mathbb{P}_1(0) \mapsto \mathbb{R}^3 \times \mathcal{S}_0^3 \times \mathbb{R}$, of $(\mathbf{u}_i, Q_i, P_i)$, by letting

$$(\widehat{\mathbf{u}}_i, \widehat{Q}_i, \widehat{P}_i)(z) = \left(\frac{\mathbf{u}_i}{\varepsilon_i}, \frac{Q_i - \overline{Q}_i}{\varepsilon_i}, \frac{P_i}{\varepsilon_i}\right)(z),\ \forall z = (x,t) \in \mathbb{P}_1(0),$$

where

$$\overline{Q}_i = \frac{1}{|\mathbb{P}_1(0)|}\int_{\mathbb{P}_1(0)} Q_i$$

denotes the average of $Q_i$ over $\mathbb{P}_1(0)$. Then $(\widehat{\mathbf{u}}_i, \widehat{Q}_i, \widehat{P}_i)$ satisfies

$$\begin{cases}\int_{\mathbb{P}_1(0)} \widehat{Q}_i = 0,\\ \int_{\mathbb{P}_1(0)} (|\widehat{\mathbf{u}}_i|^3 + |\nabla \widehat{Q}_i|^3)\,\mathrm{d}x\mathrm{d}t + \Big(\int_{\mathbb{P}_1(0)} |\widehat{P}_i|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t\Big)g^2 = 1,\\ \tau^{-2}\int_{\mathbb{P}_\tau(0)} (|\widehat{\mathbf{u}}_i|^3 + |\nabla \widehat{Q}_i|^3)\,\mathrm{d}x\mathrm{d}t + \Big(\tau^{-2}\int_{\mathbb{P}_\tau(0)} |\widehat{P}_i|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t\Big)^2 > \frac{1}{2}\max\Big\{1, C_i\frac{r_i^3}{\varepsilon_i^3}\Big\},\end{cases} \tag{5.11}$$

and $(\widehat{\mathbf{u}}_i, \widehat{Q}_i, \widehat{P}_i)$ is a suitable weak solution of the following scaled Beris–Edwards equation:

$$\begin{cases}\partial_t \widehat{Q}_i + \varepsilon_i\widehat{\mathbf{u}}_i\cdot\nabla\widehat{Q}_i - [\omega(\widehat{\mathbf{u}}_i), Q_i] = \Delta\widehat{Q}_i - \frac{r_i^2}{\varepsilon_i} f_{\mathrm{bulk}}(Q_i),\\ \partial_t\widehat{\mathbf{u}}_i + \varepsilon_i\widehat{\mathbf{u}}_i\cdot\nabla\widehat{\mathbf{u}}_i + \nabla\widehat{P}_i = \Delta\widehat{\mathbf{u}}_i - \varepsilon_i\nabla\widehat{Q}_i\Delta\widehat{Q}_i + \mathrm{div}[Q_i, \Delta\widehat{Q}_i]\\ \mathrm{div}\widehat{\mathbf{u}}_i = 0.\end{cases} \tag{5.12}$$

From (5.11), we assume that there exists

$$(\widehat{\mathbf{u}}, \widehat{Q}, \widehat{P}) \in L^3(\mathbb{P}_1(0)) \times L_t^3 W_x^{1,3}(\mathbb{P}_1(0)) \times L^{\frac{3}{2}}(\mathbb{P}_1(0))$$

such that, after passing to a subsequence,

$$(\widehat{\mathbf{u}}_i, \widehat{Q}_i, \widehat{P}_i) \rightharpoonup (\widehat{\mathbf{u}}, \widehat{Q}, \widehat{P}) \text{ in } L^3(\mathbb{P}_1(0)) \times L_t^3 W_x^{1,3}(\mathbb{P}_1(0)) \times L^{\frac{3}{2}}(\mathbb{P}_1(0)).$$

It follows from (5.11) and the lower semicontinuity that

$$\int_{\mathbb{P}_1(0)} (|\widehat{\mathbf{u}}|^3 + |\nabla\widehat{Q}|^3) + \Big(\int_{\mathbb{P}_1(0)} |\widehat{P}|^{\frac{3}{2}}\Big)^2 \le 1. \tag{5.13}$$

Moreover, we claim that

$$\|\widehat{\mathbf{u}}_i\|_{L_t^\infty L_x^2(\mathbb{P}_{\frac{1}{2}}(0))\cap L_t^2 H_x^1(\mathbb{P}_{\frac{1}{2}}(0))} + \|\nabla\widehat{Q}_i\|_{L_t^\infty L_x^2(\mathbb{P}_{\frac{1}{2}}(0))\cap L_t^2 H_x^1(\mathbb{P}_{\frac{1}{2}}(0))} \le C < \infty. \tag{5.14}$$

To show (5.14), choose a cut-off function $\phi \in C_0^\infty(\mathbb{P}_1(0))$ such that

$$0 \le \phi \le 1,\ \phi \equiv 1 \text{ on } \mathbb{P}_{\frac{1}{2}}(0),\ \text{and } |\partial_t\phi| + |\nabla\phi| + |\nabla^2\phi| \le C.$$

Define

$$\phi_i(x,t) = \phi\Big(\frac{x-x_i}{r_i}, \frac{t-t_i}{r_i^2}\Big),\ \forall (x,t) \in \mathbb{R}^3 \times (0,\infty).$$

Applying Lemma 2.2 with $\phi$ replaced by $\phi_i^2$ and applying Hölder's inequality, we would arrive at

$$\begin{aligned}&\sup_{t_i-\frac{r_i^2}{4}\le t\le t_i}\int_{B_{r_i}(x_i)}(|\mathbf{u}|^2+|\Delta Q|^2)\phi_i^2\,\mathrm{d}x+\int_{\mathbb{P}_{r_i}(z_i)}(|\nabla\mathbf{u}|^2+|\nabla^2 Q|^2)\phi_i^2\,\mathrm{d}x\mathrm{d}t\\&\leqq C\Big[\int_{\mathbb{P}_{r_i}(z_i)}(|\mathbf{u}|^2+|\nabla Q|^2)|(\partial_t+\Delta)\phi_i^2|\,\mathrm{d}x\mathrm{d}t\\&\quad+\int_{\mathbb{P}_{r_i}(z_i)}(|\mathbf{u}|^2+|\nabla Q|^2+|P|)|\mathbf{u}||\nabla\phi_i^2|\,\mathrm{d}x\mathrm{d}t+\int_{\mathbb{P}_{r_i}(z_i)}|\nabla Q|^2|\nabla^2(\phi_i^2)|\\&\quad+\int_{\mathbb{P}_{r_i}(z_i)}(|\Delta Q|+|f_{\mathrm{bulk}}(Q)|)|\mathbf{u}||\nabla\phi_i^2|+|\nabla_Q f_{\mathrm{bulk}}(Q)||\nabla Q|^2\phi_i^2\,\mathrm{d}x\mathrm{d}t\Big].\end{aligned}$$

Observe that

$$\int_{\mathbb{P}_{r_i}(z_i)}|\Delta Q||\mathbf{u}||\nabla\phi_i^2|\,\mathrm{d}x\mathrm{d}t\le\frac{1}{2}\int_{\mathbb{P}_{r_i}(z_i)}|\Delta Q|^2\phi_i^2\,\mathrm{d}x\mathrm{d}t+C\int_{\mathbb{P}_{r_i}(z_i)}|\mathbf{u}|^2|\nabla\phi_i|^2\,\mathrm{d}x\mathrm{d}t.$$

Substituting this into the above inequality and performing rescaling, we obtain that

$$\begin{aligned}&\sup_{-\frac{1}{4}\le t\le 0}\int_{B_{\frac{1}{2}}(0)}(|\widehat{\mathbf{u}}_i|^2+|\Delta\widehat{Q}_i|^2)\,\mathrm{d}x+\int_{\mathbb{P}_{\frac{1}{2}}(0)}(|\nabla\widehat{\mathbf{u}}_i|^2+|\nabla^2\widehat{Q}_i|^2)\,\mathrm{d}x\mathrm{d}t\\&\leqq C\Big[\int_{\mathbb{P}_1(0)}(|\widehat{\mathbf{u}}_i|^2+|\nabla\widehat{Q}_i|^2)+(\varepsilon_i|\widehat{\mathbf{u}}_i|^2+\varepsilon_i|\nabla\widehat{Q}_i|^2+|\widehat{P}_i|)|\widehat{\mathbf{u}}_i|\,\mathrm{d}x\mathrm{d}t\Big]\\&\quad+C\Big[\int_{\mathbb{P}_1(0)}\frac{r_i^2}{\varepsilon_i}|\widehat{\mathbf{u}}_i|\,\mathrm{d}x\mathrm{d}t+r_i^2\int_{\mathbb{P}_1(0)}|\nabla\widehat{Q}_i|^2\,\mathrm{d}x\mathrm{d}t\Big]\\&\leqq C\Big(1+\frac{r_i^2}{\varepsilon_i}+r_i^2\Big)\le C.\end{aligned}\tag{5.15}$$

This yields (5.14). From (5.14), we may also assume that

$$(\widehat{\mathbf{u}}_i,\widehat{Q}_i)\rightharpoonup(\widehat{\mathbf{u}},\widehat{Q})\text{ in }L_t^2H_x^1(\mathbb{P}_{\frac{1}{2}}(0))\times L_t^2H_x^2(\mathbb{P}_{\frac{1}{2}}(0)).\tag{5.16}$$

Since $r_i\le\varepsilon_i$ and by (5.7) $|Q_i|\le M_0$ and $|f_{\mathrm{bulk}}(Q_i)|+|\nabla_Q f_{\mathrm{bulk}}(Q_i)|\le C_0$ in $\mathbb{P}_1(0)$, there exists a constant $\overline{Q}\in\mathcal{S}_0^{(3)}$, with $|\overline{Q}|\le M_0$, such that, after passing to a subsequence,

$$Q_i\to\overline{Q}\quad\text{in}\quad L^3(\mathbb{P}_{\frac{1}{2}}(0)),$$

and

$$\frac{r_i^2}{\varepsilon_i}f_{\mathrm{bulk}}(Q_i)\to 0\quad\text{in}\quad L^\infty(\mathbb{P}_{\frac{1}{2}}(0)).$$

Hence $(\widehat{\mathbf{u}},\widehat{Q},\widehat{P}):\mathbb{P}_{\frac{1}{2}}(0)\mapsto\mathbb{R}^3\times\mathcal{S}_0^{(3)}\times\mathbb{R}$ solves the linear system:

$$\begin{cases}\partial_t\widehat{Q}-\Delta\widehat{Q}=[\omega(\widehat{\mathbf{u}}),\overline{Q}],\\\partial_t\widehat{\mathbf{u}}-\Delta\widehat{\mathbf{u}}+\nabla\widehat{P}=\mathrm{div}([\overline{Q},\Delta\widehat{Q}]),\\\mathrm{div}\widehat{\mathbf{u}}=0,\end{cases}\tag{5.17}$$

Applying Lemma 5.2 and (5.13), we know that

$$(\widehat{\mathbf{u}}, \widehat{Q}) \in C^\infty(\mathbb{P}_{\frac{1}{4}}), \ \ \widehat{P} \in L^\infty([-(\frac{1}{4})^2, 0], C^\infty(B_{\frac{1}{4}}(0)))$$

satisfies

$$\begin{aligned} &\tau^{-2} \int_{\mathbb{P}_\tau(0)} (|\widehat{\mathbf{u}}|^3 + |\nabla \widehat{Q}|^3) \, \mathrm{d}x\mathrm{d}t + \big(\tau^{-2} \int_{\mathbb{P}_\tau(0)} |\widehat{P}|^{\frac{3}{2}} \, \mathrm{d}x\mathrm{d}t\big)^2 \\ &\quad \le C\tau^3 \int_{\mathbb{P}_{\frac{1}{2}}(0)} (|\widehat{\mathbf{u}}|^3 + |\nabla \widehat{Q}|^3) \, \mathrm{d}x\mathrm{d}t + \big(\int_{\mathbb{P}_1(0)} |\widehat{P}|^{\frac{3}{2}}\big)^2 \\ &\quad \le C\tau^3, \ \forall \, \tau \in (0, \frac{1}{8}). \end{aligned} \tag{5.18}$$

We now claim that

$$(\widehat{\mathbf{u}}_i, \nabla \widehat{Q}_i) \to (\widehat{\mathbf{u}}, \nabla \widehat{Q}) \text{ in } L^3(\mathbb{P}_{\frac{3}{8}}(0)). \tag{5.19}$$

To prove (5.19), first observe that (5.15) and the equation (5.12) imply that

$$\partial_t \widehat{\mathbf{u}}_i \in \big(L_t^2 H^{-1} + L_t^2 L_x^{\frac{6}{5}} + L_t^{\frac{3}{2}} W_x^{-1, \frac{3}{2}}\big)\big(\mathbb{P}_{\frac{3}{8}}(0)\big); \ \partial_t \widehat{Q}_i \in L_t^{\frac{3}{2}} L_x^{\frac{3}{2}}(\mathbb{P}_{\frac{3}{8}}(0)),$$

enjoy the following uniform bounds:

$$\begin{aligned} &\|\partial_t \widehat{\mathbf{u}}_i\|_{\big(L_t^2 H_x^{-1} + L_t^2 L_x^{\frac{6}{5}} + L_t^{\frac{3}{2}} W_x^{-1, \frac{3}{2}}\big)(\mathbb{P}_{\frac{3}{8}}(0))} \\ &\le C\big[\|\widehat{\mathbf{u}}_i\|_{L_t^\infty L_x^2(\mathbb{P}_{\frac{1}{2}}(0))} + \|\nabla \widehat{\mathbf{u}}_i\|_{L_t^2 L_x^2(\mathbb{P}_{\frac{1}{2}}(0))} + \|\nabla \widehat{Q}_i\|^2_{L^3(\mathbb{P}_{\frac{1}{2}}(0))} + \|\nabla^2 \widehat{Q}_i\|_{L^2(\mathbb{P}_{\frac{1}{2}}(0))}\big] \\ &\le C, \end{aligned}$$

and

$$\begin{aligned} &\|\partial_t \widehat{Q}_i\|_{L^{\frac{3}{2}}(\mathbb{P}_{\frac{3}{8}}(0))} \\ &\le C\big[\|\widehat{Q}_i\|_{L_t^2 H_x^1(\mathbb{P}_{\frac{1}{2}}(0))} + \|\nabla \widehat{\mathbf{u}}_i\|_{L^2(\mathbb{P}_{\frac{1}{2}}(0))} + \|\nabla \widehat{Q}_i\|_{L^3(\mathbb{P}_{\frac{1}{2}}(0))} + \|\widehat{\mathbf{u}}_i\|_{L^3(\mathbb{P}_{\frac{1}{2}}(0))}\big] \\ &\le C. \end{aligned}$$

Thus we can apply Aubin-Lions' compactness Lemma to conclude the $L^3$-strong convergence as in (5.19).

It follows from the $L^3$-strong convergence property (5.19) that for any $\tau \in (0, \frac{1}{8})$,

$$\begin{aligned} &\tau^{-2} \int_{\mathbb{P}_\tau(0)} (|\widehat{\mathbf{u}}_i|^3 + |\nabla \widehat{Q}_i|^3) \\ &\quad = \tau^{-2} \int_{\mathbb{P}_\tau(0)} (|\widehat{\mathbf{u}}|^3 + |\nabla \widehat{Q}|^3) + \tau^{-2} o(1) \le C\tau^3 + \tau^{-2} o(1), \end{aligned} \tag{5.20}$$

where $o(1)$ stands for a quantity such that $\lim\limits_{i \to \infty} o(1) = 0$.

Now we need to estimate the pressure $\widehat{P}_i$. First, by taking divergence of the second equation $(5.8)_2$, we see that $\widehat{P}_i$ solves

$$\Delta \widehat{P}_i = -\epsilon_i \mathrm{div}^2\big[\widehat{\mathbf{u}}_i \otimes \widehat{\mathbf{u}}_i + (\nabla \widehat{Q}_i \otimes \nabla \widehat{Q}_i - \frac{1}{2}|\nabla \widehat{Q}_i|^2 I_3)\big] \text{ in } B_1, \qquad (5.21)$$

where we have applied Lemma 2.3 to guarantee that

$$\mathrm{div}^2[Q_i, \Delta \widehat{Q}_i] = 0 \ \text{ in } \ B_1.$$

We need to show that

$$\tau^{-2}\int_{\mathbb{P}_\tau(0)} |\widehat{P}_i|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t \le C\tau^{-2}(\varepsilon_i + o(1)) + C\tau,\ \forall i \ge 1. \qquad (5.22)$$

To prove (5.22), let $\eta \in C_0^\infty(B_1(0))$ be a cut-off function such that $\eta \equiv 1$ in $B_{\frac{3}{8}}(0)$, $0 \le \eta \le 1$. For any $-(\frac{3}{8})^2 \le t \le 0$, define $\widehat{P}_i^{(1)}(\cdot, t) : \mathbb{R}^3 \to \mathbb{R}$ by letting

$$\widehat{P}_i^{(1)}(x,t) = \int_{\mathbb{R}^3} \nabla_x^2 G(x-y)\eta(y)\varepsilon_i[\widehat{\mathbf{u}}_i \otimes \widehat{\mathbf{u}}_i + (\nabla \widehat{Q}_i \otimes \nabla \widehat{Q}_i - \frac{1}{2}|\nabla \widehat{Q}_i|^2 I_3)](y,t)\,\mathrm{d}y, \qquad (5.23)$$

where $G(\cdot)$ is the fundamental solution of $-\Delta$ in $\mathbb{R}^3$. Then it is easy to check that $\widehat{P}_i^{(2)}(\cdot,t) = (\widehat{P}_i - \widehat{P}_i^{(1)})(\cdot,t)$ satisfies

$$-\Delta \widehat{P}_i^{(2)}(\cdot,t) = 0 \ \text{ in } \ B_{\frac{3}{8}}(0). \qquad (5.24)$$

For $\widehat{P}_i^{(1)}$, we can apply the Calderon-Zygmund theory to show that

$$\big\|\widehat{P}_i^{(1)}\big\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \le C\epsilon_i\big[\|\widehat{\mathbf{u}}_i\|^2_{L^3(B_1(0))} + \|\nabla \widehat{Q}_i\|^2_{L^3(B_1(0))}\big] \qquad (5.25)$$

so that

$$\big\|\widehat{P}_i^{(1)}\big\|_{L^{\frac{3}{2}}(\mathbb{P}_{\frac{1}{3}}(0))} \le C\varepsilon_i(\|\widehat{\mathbf{u}}_i\|^2_{L^3(\mathbb{P}_1(0))} + \|\nabla \widehat{Q}_i\|^2_{L^3(\mathbb{P}_1(0))}) \le C(\varepsilon_i + o(1)). \qquad (5.26)$$

From the standard theory on harmonic functions, $\widehat{P}_i^{(2)}(\cdot,t) \in C^\infty(B_{\frac{1}{2}}(0))$ satisfies that for any $0 < \tau < \frac{1}{4}$,

$$\tau^{-2}\int_{\mathbb{P}_\tau(0)} |\widehat{P}_i^{(2)}|^{\frac{3}{2}} \le C\tau \int_{\mathbb{P}_{\frac{1}{3}}(0)} |\widehat{P}_i^{(2)}|^{\frac{3}{2}} \le C\tau\Big[\int_{\mathbb{P}_{\frac{1}{3}}(0)} \big(|\widehat{P}_i|^{\frac{3}{2}} + |\widehat{P}_i^{(1)}|^{\frac{3}{2}}\big) \le C\tau(1+\varepsilon_i + o(1)). \qquad (5.27)$$

Putting (5.26) and (5.27) together, we obtain (5.22).

It follows from (5.20) and (5.22) that there exist sufficiently small $\tau_0 \in (0, \frac{1}{4})$ and sufficiently large $i_0$, depending on $\tau_0$, such that for any $i \ge i_0$, it holds that

$$\tau_0^{-2}\int_{\mathbb{P}_{\tau_0}(0)} (|\widehat{\mathbf{u}}_i|^3 + |\nabla \widehat{Q}_i|^3)\,\mathrm{d}x\mathrm{d}t + \big(\tau_0^{-2}\int_{\mathbb{P}_{\tau_0}(0)} |\widehat{P}_i|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t)^2 \le \frac{1}{4}.$$

This contradicts (5.11). The proof of Lemma 5.1 is completed. □

We now need to establish the smoothness of the limit equation (5.17), namely, are have

**Lemma 5.2.** *Assume that* $(\widehat{\mathbf{u}}, \widehat{Q}) \in (L_t^\infty L_x^2 \cap L_t^2 H_x^1)(\mathbb{P}_{\frac{1}{2}}) \times (L_t^\infty H_x^1 \cap L_t^2 H_x^2)(\mathbb{P}_{\frac{1}{2}})$ *and* $\widehat{P} \in L^{\frac{3}{2}}(\mathbb{P}_{\frac{1}{2}})$ *is a weak solution of the linear system* (5.17)*, then* $(\widehat{\mathbf{u}}, \widehat{Q}) \in C^\infty(\mathbb{P}_{\frac{1}{4}})$*, and the estimate*

$$\theta^{-2} \int_{\mathbb{P}_\theta} (|\widehat{\mathbf{u}}|^3 + |\nabla \widehat{Q}|^3 + |\widehat{P}|^{\frac{3}{2}}) \le C\theta^3 \int_{\mathbb{P}_{\frac{1}{2}}} (|\widehat{\mathbf{u}}|^3 + |\nabla \widehat{Q}|^3 + |\widehat{P}|^{\frac{3}{2}}) \quad (5.28)$$

*holds for any* $\theta \in (0, \frac{1}{8})$.

*Proof.* The regularity of the limit equation (5.17) doesn't follow from the standard theory of linear parabolic equations in [20], since the source term $\mathrm{div}(\overline{Q}\Delta\widehat{Q} - \Delta\widehat{Q}\overline{Q})$ in the second equation of (5.17) depends on third order derivatives of $\widehat{Q}$. This is based on higher order energy methods, for which the cancellation property, as in the derivation of local energy inequality for suitable weak solutions of (1.6), plays a critical role.

For nonnegative multiple indices $\alpha$, $\beta$, and $\gamma$ such that $\alpha = \beta + \gamma$ and $\gamma$ is of order 1, it is easy to see that $(\nabla^\alpha \widehat{Q}, \nabla^\beta \widehat{\mathbf{u}}, \nabla^\beta \widehat{P})$ satisfies

$$\begin{cases} \partial_t(\nabla^\alpha \widehat{Q}) - \Delta(\nabla^\alpha \widehat{Q}) = [\omega(\nabla^\alpha \widehat{\mathbf{u}}), \overline{Q}], \\ \partial_t(\nabla^\beta \widehat{\mathbf{u}}) - \Delta(\nabla^\beta \widehat{\mathbf{u}}) + \nabla(\nabla^\beta \widehat{P}) = \mathrm{div}[\overline{Q}, \Delta(\nabla^\beta \widehat{Q})], \\ \mathrm{div}(\nabla^\beta \widehat{\mathbf{u}}) = 0. \end{cases} \quad (5.29)$$

Now we want to derive an arbitrarily higher order local energy inequality for (5.29). For any given $\phi \in C_0^\infty(\mathbb{P}_{\frac{1}{2}}(0))$, multiplying the first equation of (5.29) by $\nabla^\alpha \widehat{Q}\phi^2$ and integrating over $\mathbb{R}^3$, we obtain that by summing over all $\gamma$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \frac{1}{2} |\nabla(\nabla^\beta \widehat{Q})|^2 \phi^2 + \int_{\mathbb{R}^3} |\nabla^2(\nabla^\beta \widehat{Q})|^2 \phi^2 \\ &\quad = \int_{\mathbb{R}^3} \frac{1}{2} |\nabla(\nabla^\beta \widehat{Q})|^2 (\partial_t + \Delta)\phi^2 \\ &\qquad + \int_{\mathbb{R}^3} [\overline{Q}, \omega(\nabla^\beta \widehat{\mathbf{u}})] : (\Delta(\nabla^\beta \widehat{Q})\phi^2 + \nabla(\nabla^\beta \widehat{Q}) \cdot \nabla\phi^2). \end{aligned} \quad (5.30)$$

Mean while, by multiplying the second equation of (5.17) by $\nabla^\beta \widehat{\mathbf{u}}\phi^2$ and integrating over $\mathbb{R}^3$, we obtain that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \frac{1}{2} |\nabla^\beta \widehat{\mathbf{u}}|^2 \phi^2 + \int_{\mathbb{R}^3} |\nabla(\nabla^\beta \widehat{\mathbf{u}})|^2 \phi^2 \\ &\quad = \int_{\mathbb{R}^3} \frac{1}{2} |\nabla^\beta \widehat{\mathbf{u}}|^2 (\partial_t + \Delta)\phi^2 + \int_{\mathbb{R}^3} \nabla^\beta \widehat{P} \nabla^\beta \widehat{\mathbf{u}} \cdot \nabla\phi^2 \\ &\qquad + \int_{\mathbb{R}^3} [\overline{Q}, \Delta(\nabla^\beta \widehat{Q})] : (\nabla(\nabla^\beta \widehat{\mathbf{u}})\phi^2 + \nabla^\beta \widehat{\mathbf{u}} \otimes \nabla\phi^2). \end{aligned} \quad (5.31)$$

As in the above, we observe that

$$\int_{\mathbb{R}^3} [[\overline{Q}, \omega(\nabla^\beta \widehat{\mathbf{u}})] : \Delta(\nabla^\beta \widehat{Q})\phi^2 + [\overline{Q}, \Delta(\nabla^\beta \widehat{Q})] : \nabla(\nabla^\beta \widehat{\mathbf{u}})\phi^2] = 0.$$

By integration by parts we have that

$$\int_{\mathbb{R}^3} \nabla^\beta \widehat{P} \nabla^\beta \widehat{\mathbf{u}} \cdot \nabla \phi^2 = (-1)^{|\beta|} \int_{\mathbb{R}^3} \widehat{\mathbf{u}} \cdot \nabla^\beta (\nabla^\beta \widehat{P} \nabla \phi^2). \tag{5.32}$$

It follows from the second equation of (5.17) that $\widehat{P}$ solves

$$\Delta \widehat{P} = \operatorname{div}^2 [\overline{Q}, \Delta \widehat{Q}] = 0, \text{ in } B_{\frac{1}{2}(0)},$$

where we have applied Lemma 2.3. Hence, by the standard regularity theory of harmonic functions,

$$\int_{B_{\frac{3}{8}(0)}} |\nabla^l \widehat{P}|^{\frac{3}{2}} \leq C \int_{B_{\frac{1}{2}}(0)} |\widehat{P}|^{\frac{3}{2}}, \quad l = k, k+1, ..., 2k, \tag{5.33}$$

so that, by Young's inequality, we can derive from (5.32) and (5.33) that

$$\left| \int_{\mathbb{R}^3} \nabla^\beta \widehat{P} \nabla^\beta \widehat{\mathbf{u}} \cdot \nabla \phi^2 \right| \leq C \int_{B_{\frac{1}{2}}(0)} (|\widehat{\mathbf{u}}|^3 + |\widehat{P}|^{\frac{3}{2}}).$$

Hence, by adding (5.30) and (5.31) together and then taking summation over all $\beta$'s with $|\beta| = k \geq 0$, we obtain that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \frac{1}{2} (|\nabla^k \widehat{\mathbf{u}}|^2 + |\nabla^{k+1} \widehat{Q}|^2)\phi^2 + \int_{\mathbb{R}^3} (|\nabla^{k+1} \widehat{\mathbf{u}}|^2 + |\nabla^{k+2} \widehat{Q}|^2)\phi^2 \\
&\quad \leq \int_{\mathbb{R}^3} \frac{1}{2} (|\nabla^k \widehat{\mathbf{u}}|^2 + |\nabla^{k+1} \widehat{Q}|^2)(|\partial_t(\phi^2)| + |\nabla^2(\phi^2)|) \\
&\qquad + C \int_{B_{\frac{1}{2}}(0)} (|\widehat{\mathbf{u}}|^3 + |\widehat{P}|^{\frac{3}{2}}) \\
&\qquad + C \int_{\mathbb{R}^3} \big(|\nabla^{k+1} \widehat{\mathbf{u}}||\nabla^{k+1} \widehat{Q}| + |\nabla^k \widehat{\mathbf{u}}||\nabla^{k+2} \widehat{Q}|\big)|\nabla \phi^2| \\
&\quad \leq \int_{\mathbb{R}^3} \frac{1}{2} (|\nabla^k \widehat{\mathbf{u}}|^2 + |\nabla^{k+1} \widehat{Q}|^2)(|\partial_t(\phi^2)| + |\nabla^2(\phi^2)|) \\
&\qquad + C \int_{B_{\frac{1}{2}}(0)} (|\widehat{\mathbf{u}}|^3 + |\widehat{P}|^{\frac{3}{2}}) \\
&\qquad + \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla^{k+1} \widehat{\mathbf{u}}|^2 + |\nabla^{k+2} \widehat{Q}|^2)\phi^2 + C \int_{\mathbb{R}^3} \big(|\nabla^k \widehat{\mathbf{u}}|^2 + |\nabla^{k+1} \widehat{Q}|^2\big)|\nabla \phi|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(|\nabla^k\widehat{\mathbf{u}}|^2+|\nabla^{k+1}\widehat{Q}|^2)\phi^2+\int_{\mathbb{R}^3}(|\nabla^{k+1}\widehat{\mathbf{u}}|^2+|\nabla^{k+2}\widehat{Q}|^2)\phi^2\\
&\le C\int_{\mathbb{R}^3}(|\nabla^k\widehat{\mathbf{u}}|^2+|\nabla^{k+1}\widehat{Q}|^2)(|\partial_t(\phi^2)|+|\nabla^2(\phi^2)|)\\
&\quad +C\int_{B_{\frac12}(0)}(|\widehat{\mathbf{u}}|^3+|\widehat{P}|^{\frac32})\\
&\quad +C\int_{\mathbb{R}^3}\big(|\nabla^k\widehat{\mathbf{u}}|^2+|\nabla^{k+1}\widehat{Q}|^2\big)|\nabla\phi|^2. \qquad (5.34)
\end{aligned}$$

By choosing suitable test functions $\phi$, it is not hard to see that (5.34) implies that for $k\ge 0$,

$$\begin{aligned}
&\sup_{-\frac{1}{16}\le t\le 0}\int_{B_{\frac14}(0)}(|\nabla^k\widehat{\mathbf{u}}|^2+|\nabla^{k+1}\widehat{Q}|^2)+\int_{\mathbb{P}_{\frac14}(0)}(|\nabla^{k+1}\widehat{\mathbf{u}}|^2+|\nabla^{k+2}\widehat{Q}|^2\\
&\le C\int_{\mathbb{P}_{\frac38}(0)}(|\nabla^k\widehat{\mathbf{u}}|^2+|\nabla^{k+1}\widehat{Q}|^2)+C\int_{\mathbb{P}_{\frac12}(0)}(|\widehat{\mathbf{u}}|^3+|\widehat{P}|^{\frac32}). \qquad (5.35)
\end{aligned}$$

It is clear that with suitable adjustment of radius, applying (5.35) inductively on $k$ yields that

$$\begin{aligned}
&\sup_{-\frac{1}{16}\le t\le 0}\int_{B_{\frac14}(0)}(|\nabla^k\widehat{\mathbf{u}}|^2+|\nabla^{k+1}\widehat{Q}|^2)+\int_{\mathbb{P}_{\frac14}(0)}(|\nabla^{k+1}\widehat{\mathbf{u}}|^2+|\nabla^{k+2}\widehat{Q}|^2)\\
&\le C\int_{\mathbb{P}_{\frac38}(0)}(|\nabla\widehat{\mathbf{u}}|^2+|\nabla^2\widehat{Q}|^2+C\int_{\mathbb{P}_{\frac12}(0)}(|\widehat{\mathbf{u}}|^3+|\widehat{P}|^{\frac32}),\ \forall k\ge 1. \qquad (5.36)
\end{aligned}$$

With (5.36), we can apply the regularity theory for both the linear Stokes equation and the linear parabolic equation to conclude that $(\widehat{\mathbf{u}},\widehat{Q})\in C^\infty(\mathbb{P}_{\frac14}(0))$. Furthermore, applying the elliptic estimate for the pressure equation (5.21) we see that $\nabla^k\widehat{P}\in C^0(\mathbb{P}_{\frac14}(0))$ for any $k\ge 1$. For $l\ge 1$, taking $t$-derivative $\partial_t^l$ of both sides of (5.21), we can also see that $\nabla^k\partial_t^l\widehat{P}\in C^0(\mathbb{P}_{\frac14}(0))$. Therefore $(\widehat{\mathbf{u}},\widehat{Q},\widehat{P})\in C^\infty(\mathbb{P}_{\frac14}(0))$ and the estimate (5.28) holds. This completes the proof of Lemma 5.2. □

Now we can iterate Lemma 5.1 and utilize the Riesz potential estimates in Morrey spaces to obtain the following $\varepsilon_0$-regularity:

**Lemma 5.3.** *For any $M>0$, there exists $\varepsilon_0>0$, depending on $M$, such that if $(\mathbf{u},Q,P)$ is a suitable weak solution of (1.6) in $\Omega\times(0,\infty)$, which satisfies, for $z_0=(x_0,t_0)\in\Omega\times(r_0^2,\infty)$ and*

$$\begin{cases}|Q|\le M & \text{if } F_{\text{bulk}}=F_{\text{LdG}} \text{ and } \Omega=\mathbb{R}^3,\\ |G_{\text{BM}}(Q)|\le M & \text{if } F_{\text{bulk}}=F_{\text{BM}} \text{ and } \Omega=\mathbb{T}^3,\end{cases}\quad \text{in } \mathbb{P}_{r_0}(z_0), \qquad (5.37)$$

*and*

$$r_0^{-2}\int_{\mathbb{P}_{r_0}(z_0)}(|\mathbf{u}|^3+|\nabla Q|^3)\,\mathrm{d}x\mathrm{d}t+\Big(r_0^{-2}\int_{\mathbb{P}_{r_0}(z_0)}|P|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t\Big)^2\le\varepsilon_0^3, \quad (5.38)$$

*then for any* $1<p<\infty$, $(\mathbf{u},P,\nabla Q)\in L^p(\mathbb{P}_{\frac{r_0}{4}}(z_0))$ *and*

$$\big\|(\mathbf{u},P,\nabla Q)\big\|_{L^p(\mathbb{P}_{\frac{r_0}{4}}(z_0))}\le C(p,\varepsilon_0,M). \quad (5.39)$$

*Proof.* From (5.38), we have that

$$\Big(\frac{r_0}{2}\Big)^{-2}\int_{\mathbb{P}_{\frac{r_0}{2}}(z)}(|\mathbf{u}|^3+|\nabla Q|^3)\,\mathrm{d}x\mathrm{d}t+\Big(\Big(\frac{r_0}{2}\Big)^{-2}\int_{\mathbb{P}_{\frac{r_0}{2}}(z)}|P|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t\Big)^2\le 8\varepsilon_0^3 \quad (5.40)$$

holds for any $z\in\mathbb{P}_{\frac{r_0}{2}}(z_0)$. By applying Lemma 5.1 repeatedly on $\mathbb{P}_{\frac{r_0}{2}}(z)$ for $z\in\mathbb{P}_{\frac{r_0}{2}}(z_0)$, we have that $C_0>0$ and $\tau_0\in(0,\frac{1}{2})$ such that for any $k\ge 1$,

$$\begin{aligned}&(\tau_0^k r_0)^{-2}\int_{\mathbb{P}_{\tau_0^k r_0}(z)}(|\mathbf{u}|^3+|\nabla Q|^3)\,\mathrm{d}x\mathrm{d}t+\big((\tau_0^k r_0)^{-2}\int_{\mathbb{P}_{\tau_0^k r_0}(z)}|P|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t)^2\\&\le 2^{-k}\max\Bigg\{\Big(\frac{r_0}{2}\Big)^{-2}\int_{\mathbb{P}_{\frac{r_0}{2}}(z)}(|\mathbf{u}|^3+|\nabla Q|^3)\,\mathrm{d}x\mathrm{d}t\\&\qquad+\Bigg(\Big(\frac{r_0}{2}\Big)^{-2}\int_{\mathbb{P}_{\frac{r_0}{2}}(z)}|P|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t\Bigg)^2,\ \frac{C_0r_0^3}{1-2\tau_0^3}\Bigg\}. \end{aligned} \quad (5.41)$$

Therefore, for $\theta_0=\frac{\ln 2}{3|\ln\tau_0|}\in(0,\frac{1}{3})$, it holds that for any $0<s<\frac{r_0}{2}$ and $z\in\mathbb{P}_{\frac{r_0}{2}}(z_0)$,

$$s^{-2}\int_{\mathbb{P}_s(z)}(|\mathbf{u}|^3+|\nabla Q|^3+|P|^{\frac{3}{2}})\,\mathrm{d}x\mathrm{d}t\le C(1+\varepsilon_0^3)\Big(\frac{s}{r_0}\Big)^{3\theta_0}. \quad (5.42)$$

By (5.37) and Lemma 3.2, there exists $C>0$, depending on $M$, such that

$$|Q|+|f_{\text{bulk}}(Q)|+|\nabla_Q f_{\text{bulk}}(Q)|\le C \text{ in } \mathbb{P}_{r_0}(z_0). \quad (5.43)$$

Now we can apply the local energy inequality (1.12) for $(\mathbf{u},P,Q)$ on $\mathbb{P}_{\frac{r_0}{2}}(z)$, for $z\in\mathbb{P}_{\frac{r_0}{2}}(z_0)$, to get that for $0<s<\frac{r_0}{2}$,

$$\begin{aligned}&s^{-1}\int_{\mathbb{P}_s(z)}(|\nabla u|^2+|\Delta Q|^2)\,\mathrm{d}x\mathrm{d}t\\&\le C\Big[(2s)^{-3}\int_{\mathbb{P}_{2s}(z)}(|u|^2+|\nabla Q|^2)+(2s)^{-2}\int_{\mathbb{P}_{2s}(z)}(|u|^3+|\nabla Q|^3+|P|^{\frac{3}{2}})\\&\quad+(2s)^{-2}\int_{\mathbb{P}_{2s}(z)}|u|+(2s)^{-1}\int_{\mathbb{P}_{2s}(z)}|\nabla Q|^2\Big]\\&\le C(1+\varepsilon_0^3)\Big(\frac{s}{r_0}\Big)^{2\theta_0}.\end{aligned} \quad (5.44)$$

Next we employ the estimate of Riesz potentials in Morrey spaces to prove the smoothness of $(\mathbf{u}, P, Q)$ near $z_0$, analogous to that by Huang–Wang [19], Hineman–Wang [17], and Huang–Lin–Wang [18].

For any open set $U \subset \mathbb{R}^3 \times \mathbb{R}$, $1 \le p < \infty$, and $0 \le \lambda \le 5$, define the Morrey space $M^{p,\lambda}(U)$ by

$$M^{p,\lambda}(U) := \Big\{ f \in L^p_{\mathrm{loc}}(U) : \ \|f\|^p_{M^{p,\lambda}(U)} = \sup_{z\in U, r>0} r^{\lambda-5} \int_{\mathbb{P}_r(z)} |f|^p \,\mathrm{d}x\mathrm{d}t < \infty \Big\}.$$

It follows from (5.42) and (5.44) that there exists $\alpha \in (0, 1)$ such that

$$(\mathbf{u}, \nabla Q) \in M^{3,3(1-\alpha)}\big(\mathbb{P}_{\frac{r_0}{2}}(z_0)\big), \ \ P \in M^{\frac{3}{2},3(1-\alpha)}\big(\mathbb{P}_{\frac{r_0}{2}}(z_0)\big),$$
$$(\nabla\mathbf{u}, \nabla^2 Q) \in M^{2,4-2\alpha}\big(\mathbb{P}_{\frac{r_0}{2}}(z_0)\big).$$

Write $(3.1)_1$ as

$$\partial_t Q - \Delta Q = f, \ \ f \equiv -\mathbf{u}\cdot\nabla Q + [\omega, Q] - f_{\mathrm{bulk}}(Q) \in M^{\frac{3}{2},3(1-\alpha)}\big(\mathbb{P}_{\frac{r_0}{2}}(z_0)\big). \tag{5.45}$$

Let $\eta \in C_0^\infty(\mathbb{R}^4)$ be a cut off function of $\mathbb{P}_{\frac{r_0}{2}}(z_0)$ such that $0 \le \eta \le 1$, $\eta = 1$ in $\mathbb{P}_{\frac{r_0}{2}}(z_0)$, $|\partial_t\eta| + |\nabla^2\eta| \le Cr_0^{-2}$, Set $w = \eta^2(Q - Q_{z_0,r_0})$, where $Q_{z_0,r_0}$ is the average of $Q$ over $\mathbb{P}_{\frac{r_0}{2}}(z_0)$. Then

$$\partial_t w - \Delta w = F, \ \ F := \eta^2 f + (\partial_t\eta^2 - \Delta\eta^2)(Q - Q_{z_0,r_0}) - \nabla\eta^2\cdot\nabla Q \tag{5.46}$$

We can check that $F \in M^{\frac{3}{2},3(1-\alpha)}(\mathbb{R}^4)$ and that it satisfies

$$\|F\|_{M^{\frac{3}{2},3(1-\alpha)}(\mathbb{R}^4)} \le C(1+\varepsilon_0). \tag{5.47}$$

Let $\Gamma$ denote the heat kernel in $\mathbb{R}^3$. Then

$$|\nabla\Gamma|(x,t) \le C\delta^{-4}((x,t),(0,0)), \ \forall (x,t) \ne (0,0),$$

where $\delta(\cdot,\cdot)$ denotes the parabolic distance on $\mathbb{R}^4$. By the Duhamel formula, we have that

$$|w(x,t)| \le \int_0^t \int_{\mathbb{R}^3} |\nabla\Gamma(x-y,t-s)||F(y,s)| \,\mathrm{d}y\mathrm{d}s \le C\mathcal{I}_1(|F|)(x,t), \tag{5.48}$$

where $\mathcal{I}_\beta$ is the Riesz potential of order $\beta$ on $\mathbb{R}^4$, $\beta \in [0, 4]$, defined by

$$\mathcal{I}_\beta(g)(x,t) = \int_{\mathbb{R}^4} \frac{|g(y,s)|}{\delta^{5-\beta}((x,t),(y,s))} \,\mathrm{d}y\mathrm{d}s, \ \forall g \in L^1(\mathbb{R}^4).$$

Applying the Riesz potential estimates (see [19] Theorem 3.1), we conclude that $\nabla w \in M^{\frac{3(1-\alpha)}{1-2\alpha},3(1-\alpha)}(\mathbb{R}^4)$ and

$$\Big\|\nabla w\Big\|_{M^{\frac{3(1-\alpha)}{1-2\alpha},3(1-\alpha)}(\mathbb{R}^4)} \le C\Big\|F\Big\|_{M^{\frac{3}{2},3(1-\alpha)}(\mathbb{R}^4)} \le C(1+\varepsilon_0). \tag{5.49}$$

Since $\lim_{\alpha\uparrow\frac{1}{2}} \frac{3(1-\alpha)}{1-2\alpha} = \infty$, we conclude that for any $1 < p < \infty$, $\nabla w \in L^p(\mathbb{P}_{r_0}(z_0))$ and

$$\left\|\nabla w\right\|_{L^p(\mathbb{P}_{r_0}(z_0))} \leq C(p, r_0, \varepsilon_0). \tag{5.50}$$

Since $Q - w$ solves

$$\partial_t(Q-w) - \Delta(Q-w) = 0 \quad \text{in} \quad \mathbb{P}_{\frac{r_0}{2}}(z_0),$$

it follows from the theory of heat equations that for any $1 < p < \infty$, $\nabla Q \in \mathbb{P}_{\frac{r_0}{2}}(z_0)$ and

$$\left\|\nabla Q\right\|_{L^p(\mathbb{P}_{\frac{r_0}{2}}(z_0))} \leq C(p, r_0, \varepsilon_0). \tag{5.51}$$

We now proceed with the estimation of $\mathbf{u}$. Let $\mathbf{v} : \mathbb{R}^3 \times (0, \infty) \mapsto \mathbb{R}^3$ solve the Stokes equation

$$\begin{cases} \partial_t \mathbf{v} - \Delta \mathbf{v} + \nabla P & \\ = -\mathrm{div}\big[\eta^2\big(\mathbf{u}\otimes\mathbf{u} + (\nabla Q \otimes \nabla Q - \frac{1}{2}|\nabla Q|^2 I_3)\big)\big] + \mathrm{div}\big[\eta^2[Q, \Delta Q]\big] & \text{in } \mathbb{R}^4_+, \\ \mathrm{div}\mathbf{v} = 0 & \text{in } \mathbb{R}^4_+, \\ \mathbf{v}(\cdot, 0) = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{5.52}$$

By using the Oseen kernel (see Leray [21]), an estimate of $\mathbf{v}$ can be given by

$$|\mathbf{v}(x,t)| \leq C\mathcal{I}_1(|X|)(x,t), \ \forall (x,t) \in \mathbb{R}^3 \times (0, \infty), \tag{5.53}$$

where

$$X = \eta^2\big[\mathbf{u}\otimes\mathbf{u} + (\nabla Q \otimes \nabla Q - \frac{1}{2}|\nabla Q|^2 I_3) + [Q, \Delta Q]\big].$$

As above, we can check that $X \in M^{\frac{3}{2}, 3(1-\alpha)}(\mathbb{R}^4)$ and

$$\begin{aligned} \left\|X\right\|_{M^{\frac{3}{2},3(1-\alpha)}(\mathbb{R}^4)} &\leq C\Big[\|\mathbf{u}\|^2_{M^{3,3(1-\alpha)}(\mathbb{P}_{\frac{r_0}{2}}(z_0))} + \|\nabla Q\|^2_{M^{3,3(1-\alpha)}(\mathbb{P}_{\frac{r_0}{2}}(z_0))} \\ &\quad + \|\Delta Q - f_{\mathrm{bulk}}(Q)\|_{M^{3,3(1-\alpha)}(\mathbb{P}_{\frac{r_0}{2}}(z_0))}\Big] \\ &\leq C(1+\varepsilon_0). \end{aligned}$$

Hence we conclude that $\mathbf{v} \in M^{\frac{3(1-\alpha)}{1-2\alpha}, 3(1-\alpha)}(\mathbb{R}^4)$ and

$$\left\|\mathbf{v}\right\|_{M^{\frac{3(1-\alpha)}{1-2\alpha},3(1-\alpha)}(\mathbb{R}^4)} \leq C\left\|X\right\|_{M^{\frac{3}{2},3(1-\alpha)}(\mathbb{R}^4)} \leq C(1+\varepsilon_0). \tag{5.54}$$

As $\alpha \uparrow \frac{1}{2}$, we conclude that for any $1 < p < \infty$, $\mathbf{v} \in L^p(\mathbb{P}_{r_0}(z_0))$ and

$$\left\|\mathbf{v}\right\|_{L^p(\mathbb{P}_{r_0}(z_0))} \leq C(p, r_0, \varepsilon_0). \tag{5.55}$$

Note that $\mathbf{u}-\mathbf{v}$ solves the linear homogeneous Stokes equation in $\mathbb{P}_{\frac{r_0}{2}}(z_0)$:

$$\partial_t(\mathbf{u}-\mathbf{v})-\Delta(\mathbf{u}-\mathbf{v})+\nabla P=0,\ \operatorname{div}(\mathbf{u}-\mathbf{v})=0\ \text{ in }\ \mathbb{P}_{\frac{r_0}{2}}(z_0).$$

Then $\mathbf{u}-\mathbf{v}\in L^\infty(\mathbb{P}_{\frac{r_0}{4}}(z_0))$. Therefore for any $1<p<\infty$, $\mathbf{u}\in L^p(\mathbb{P}_{\frac{r_0}{4}}(z_0))$ and

$$\|\mathbf{u}\|_{L^p(\mathbb{P}_{\frac{r_0}{4}}(z_0))}\le C(p,r_0,\varepsilon_0). \tag{5.56}$$

For $P$, since it satisfies the Poisson equation, for $t_0-\frac{r_0^2}{4}\le t\le t_0$,

$$-\Delta P=\operatorname{div}^2\big[\mathbf{u}\otimes\mathbf{u}+(\nabla Q\otimes\nabla Q-\frac{1}{2}|\nabla Q|^2 I_3)\big]\ \text{ in }\ B_{\frac{r_0}{2}}(x_0). \tag{5.57}$$

Hence $P\in L^p(\mathbb{P}_{\frac{r_0}{4}}(z_0))$ and satisfies the (5.39). The proof is now complete. □

The higher order regularity of (3.1) does not follow from the standard theory, since the equation for $\mathbf{u}$ involves $\nabla^3 Q$ and the equation for $Q$ involves $\nabla\mathbf{u}$. It turns out that the higher order regularity of (3.1) can be obtained through higher oder energy methods. Roughly speaking, if $(\mathbf{u},P,\nabla Q)$ is in $L^p$ for any $1<p<\infty$, then (3.1) can be viewed as a perturbed version of the linear equation (5.17) with controllable error terms. Here higher order versions of the cancellation properties (1.13) and (1.16) in the local energy inequality (1.12) also plays an important role. This kind of idea has been previously employed by Huang-Lin-Wang (see [18] Lemma 3.4) for general Ericksen-Leslie systems in dimension two. More precisely, we have

**Lemma 5.4.** *Under the same assumptions as Lemma 5.3, we have that for any* $k\ge 0$, $(\nabla^k\mathbf{u},\nabla^{k+1}Q)\in\big(L_t^\infty L_x^2\cap L_t^2 H_x^1\big)(\mathbb{P}_{\frac{1+2^{-(k+1)}}{2}r_0}(z_0))$ *and the following estimates hold:*

$$\begin{aligned}&\sup_{t_0-\left(\frac{(1+2^{-(k+1)})}{2}r_0\right)^2\le t\le t_0}\int_{B_{\frac{1+2^{-(k+1)}}{2}r_0}(x_0)}(|\nabla^k\mathbf{u}|^2+|\nabla^{k+1}Q|^2)\,\mathrm{d}x\\&\quad+\int_{\mathbb{P}_{\frac{1+2^{-(k+1)}}{2}r_0}(z_0)}(|\nabla^{k+1}\mathbf{u}|^2+|\nabla^{k+2}Q|^2+|\nabla^k P|^{\frac{5}{3}})\,\mathrm{d}x\mathrm{d}t\\&\quad\le C(k,r_0,)\varepsilon_0.\end{aligned} \tag{5.58}$$

*In particular,* $(\mathbf{u},Q)$ *is smooth in* $\mathbb{P}_{\frac{r_0}{4}}(z_0)$.

*Proof.* For simplicity, assume $z_0=(0,0)$ and $r_0=8$. (5.58) can be proved by an induction on $k$. It is clear that when $k=0$, (5.58) follows directly from the local energy inequality (1.12). Here we indicate how to prove (5.58) for $k=1$. First, recall from Lemma 5.3 that, for any $i\in\mathbb{N}^+$ and $1<p<\infty$,

$$\begin{aligned}&\|Q\|_{L^\infty(\mathbb{P}_2)}+\|\nabla^i f_{\text{bulk}}(Q)\|_{L^\infty(\mathbb{P}_2)}\\&\quad\le C(i,\varepsilon_0),\ \|(\mathbf{u},P,\nabla Q)\|_{L^p(\mathbb{P}_2)}\le C(p)\varepsilon_0.\end{aligned} \tag{5.59}$$

Taking the spatial derivative of (1.6)[1], we have

$$\begin{cases} \partial_t Q_\alpha + \mathbf{u}\cdot\nabla Q_\alpha + \mathbf{u}_\alpha\cdot\nabla Q - [\omega_\alpha, Q] - [\omega, Q_\alpha] \\ = \Delta Q_\alpha - (f_{\text{bulk}}(Q))_\alpha, \\ \partial_t \mathbf{u}_\alpha + \mathbf{u}\cdot\nabla\mathbf{u}_\alpha + \mathbf{u}_\alpha\cdot\nabla\mathbf{u} + \nabla P_\alpha \\ = \Delta\mathbf{u}_\alpha - \nabla Q\cdot\Delta Q_\alpha - \nabla Q_\alpha\cdot\Delta Q + \operatorname{div}[Q, \Delta Q]_\alpha, \\ \operatorname{div}\mathbf{u}_\alpha = 0, \end{cases} \quad \text{in } \mathbb{P}_1. \tag{5.60}$$

Here $\omega_\alpha = \omega(\mathbf{u}_\alpha)$. Let $\eta \in C_0^\infty(B_2)$ be such that

$$0 \le \eta \le 1,\ \ \eta \equiv 1 \text{ in } B_{1+2^{-2}},\ \ \eta \equiv 0 \text{ out } B_{1+2^{-1}},\ \ |\nabla\eta| + |\nabla^2\eta| \le 16.$$

Taking $\nabla$ of $(5.60)_1$ and multiplying it by $\nabla Q_\alpha\eta^2$, and multiplying $(5.60)_2$ by $\nabla\mathbf{u}_\alpha\eta^2$, and then, integrating the resulting equations over $B_2$[2], we obtain that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla^2 Q|^2\eta^2 - \int_{\mathbb{R}^3}(\mathbf{u}_\alpha\cdot\nabla)Q\cdot\Delta Q_\alpha\eta^2 - \int_\Omega(\mathbf{u}\cdot\nabla)Q_\alpha\cdot(\Delta Q_\alpha\eta^2 + \nabla Q_\alpha\nabla\eta^2) \\ &\quad - \int_\Omega(\mathbf{u}_\alpha\cdot\nabla)Q\cdot\nabla Q_\alpha\nabla\eta^2 - \int_\Omega[Q, \omega_\alpha]\cdot(\Delta Q_\alpha\eta^2 + \nabla Q_\alpha\nabla\eta^2) \\ &= \int_\Omega\big[[Q_\alpha, \omega] - (\Delta Q_\alpha - (f_{\text{bulk}}(Q))_\alpha)\big]\cdot(\Delta Q_\alpha\eta^2 + \nabla Q_\alpha\nabla\eta^2), \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\mathbf{u}|^2\eta^2 - \int_\Omega\frac{|\nabla\mathbf{u}|^2}{2}\mathbf{u}\cdot\nabla\eta^2 + \int_\Omega(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}\cdot\mathbf{u}_\alpha\eta^2 - \int_\Omega P_\alpha\mathbf{u}_\alpha\cdot\nabla\eta^2 \\ &= -\int_\Omega\Big(|\nabla^2\mathbf{u}|^2\eta^2 - \frac{|\nabla\mathbf{u}|^2}{2}\Delta\eta^2\Big) - \int_\Omega\big((\mathbf{u}_\alpha\cdot\nabla)Q\cdot\Delta Q_\alpha\eta^2 + (\mathbf{u}_\alpha\cdot\nabla)Q_\alpha\cdot\Delta Q\eta^2\big) \\ &\quad - \int_\Omega[Q_\alpha, \Delta Q]\cdot(\nabla\mathbf{u}_\alpha\eta^2 + \mathbf{u}_\alpha\otimes\nabla\eta^2) - \int_\Omega[Q, \Delta Q_\alpha]\cdot(\nabla\mathbf{u}_\alpha\eta^2 + \mathbf{u}_\alpha\otimes\nabla\eta^2). \end{aligned}$$

Adding these two equations together and regrouping terms, and using the cancellation identity

$$\int_\Omega[Q, \omega_\alpha]\cdot\Delta Q_\alpha\eta^2 = \int_\Omega[Q, \Delta Q_\alpha]\cdot\nabla\mathbf{u}_\alpha\eta^2,$$

[1] Strictly speaking, we need to take finite quotient $D_h^j$ of (1.6) ($j = 1, 2, 3$) and then send $h \to 0$.

[2] Strictly speaking, we need to multiply $\Delta(D_h^j Q)\eta^2$ and $\nabla(D_h^j\mathbf{u})\eta^2$ and then send $h \to 0$.

we arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|\nabla\mathbf{u}|^2+|\nabla^2Q|^2)\eta^2+\int_\Omega(|\nabla^2\mathbf{u}|^2+|\Delta\nabla Q|^2)\eta^2\\
&=\int_\Omega[(\mathbf{u}\cdot\nabla)Q_\alpha\cdot(\Delta Q_\alpha\eta^2+\nabla Q_\alpha\nabla\eta^2)+(\mathbf{u}_\alpha\cdot\nabla)Q\cdot\nabla Q_\alpha\nabla\eta^2]\\
&\quad+\int_\Omega([Q,\omega_\alpha]-\Delta Q_\alpha):\nabla Q_\alpha\nabla\eta^2\\
&\quad+\int_\Omega\big([Q_\alpha,\omega]+(f_{\text{bulk}}(Q))_\alpha\big):(\Delta Q_\alpha\eta^2+\nabla Q_\alpha\nabla\eta^2)\\
&\quad+\int_\Omega[\frac{|\nabla\mathbf{u}|^2}{2}(\Delta\eta^2+\mathbf{u}\cdot\nabla\eta^2)-\mathbf{u}_\alpha\cdot(\nabla\mathbf{u}\cdot\mathbf{u}_\alpha+\nabla Q_\alpha:\Delta Q)\eta^2+P_\alpha\mathbf{u}_\alpha\cdot\nabla\eta^2]\\
&\quad-\int_\Omega[Q_\alpha,\Delta Q]:(\nabla\mathbf{u}_\alpha\eta^2+\mathbf{u}_\alpha\otimes\nabla\eta^2)-\int_\Omega[Q,\Delta Q_\alpha]:\mathbf{u}_\alpha\otimes\nabla\eta^2\\
&:=\sum_{i=1}^{6}A_i.
\end{aligned}
$$

We can estimate the $A_i$'s separately as follows:

$$
\begin{aligned}
|A_6|&\le\frac{1}{16}\int_\Omega|\Delta\nabla Q|^2\eta^2+C\int_\Omega(|\nabla Q|^2\eta^2+|\nabla\mathbf{u}|^2(\eta^2+|\nabla\eta|^2),\\
|A_5|&\le\frac{1}{16}\int_\Omega|\nabla^2\mathbf{u}|^2\eta^2+C\int_\Omega|\nabla Q|^2|\Delta Q|^2\eta^2+C\int_\Omega|\nabla\mathbf{u}|^2|\nabla\eta|^2,\\
|A_4|&\le\frac{1}{8}\int_\Omega(|\nabla^2\mathbf{u}|^2+|\Delta\nabla Q|^2)\eta^2+C\int_\Omega[|\nabla\mathbf{u}|^2|\Delta\eta^2|+|\mathbf{u}|^2(|\nabla\mathbf{u}|^2+|\Delta Q|^2)\eta^2]\\
&\quad+C\int_\Omega(|\nabla\mathbf{u}|^2+|\Delta Q|^2)|\nabla\eta|^2+C\int_\Omega(|P|^2|\nabla\eta|^2+|P||\nabla\mathbf{u}||\Delta\eta^2|),\\
|A_3|&\le\frac{1}{16}\int_\Omega|\Delta\nabla Q|^2\eta^2+C\int_\Omega|\nabla Q|^2(|\nabla\mathbf{u}|^2+|\Delta Q|^2)\eta^2\\
&\quad+C\int_\Omega(|\nabla Q|^2\eta^2+|\nabla\mathbf{u}|^2|\nabla\eta|^2),\\
|A_2|&\le\frac{1}{16}\int_\Omega|\Delta\nabla Q|^2\eta^2+C\int_\Omega(|\nabla\mathbf{u}|^2+|\Delta Q|^2)|\nabla\eta|^2,\\
|A_1|&\le\frac{1}{16}\int_\Omega|\Delta\nabla Q|^2\eta^2+C\int_\Omega[(|\mathbf{u}|^2|+|\nabla Q|^2)\Delta Q|^2\eta^2+(|\nabla\mathbf{u}|^2+|\Delta Q|^2)|\nabla\eta|^2].
\end{aligned}
$$

Substituting these estimates on the $A_i$'s into the above inequality, we obtain that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|\nabla\mathbf{u}|^2+|\nabla^2Q|^2)\eta^2+\int_\Omega(|\nabla^2\mathbf{u}|^2+|\Delta\nabla Q|^2)\eta^2\\
&\quad\le C\int_{B_{1+2^{-1}}}(|\mathbf{u}|^2+|\nabla Q|^2+|\nabla\mathbf{u}|^2+|\Delta Q|^2+|P|^2)\\
&\qquad+C\int_\Omega(|\mathbf{u}|^2|\nabla\mathbf{u}|^2+|\mathbf{u}|^2|\Delta Q|^2+|\nabla Q|^2|\Delta Q|^2+|\nabla Q|^2|\nabla\mathbf{u}|^2)\eta^2.
\end{aligned}
$$

Now we want to estimate the second term in the right hand side. By Sobolev-interpolation inequalities, we have

$$
\begin{aligned}
&\int_\Omega |\mathbf{u}|^2|\nabla\mathbf{u}|^2\eta^2\\
&\quad\le \|\nabla\mathbf{u}\eta\|_{L^2(\Omega)}\|\nabla\mathbf{u}\eta\|_{L^3(\Omega)}\|\mathbf{u}\|^2_{L^{12}(B_{1+2^{-1}})}\\
&\quad\le C\|\nabla\mathbf{u}\eta\|_{L^2(\Omega)}\|\nabla\mathbf{u}\eta\|^{\frac12}_{L^2(\Omega)}\|\nabla(\nabla\mathbf{u}\eta)\|^{\frac12}_{L^2(\Omega)}\|\mathbf{u}\|^2_{L^{12}(B_{1+2^{-1}})}\\
&\quad\le C\|\nabla\mathbf{u}\eta\|_{L^2(\Omega)}\|\nabla(\nabla\mathbf{u}\eta)\|_{L^2(\Omega)}\|\mathbf{u}\|^2_{L^{12}(B_{1+2^{-1}})}\\
&\quad\le \frac18\int_\Omega|\nabla^2\mathbf{u}|^2\eta^2 + C\int_{B_{1+2^{-1}}}|\nabla\mathbf{u}|^2 + C\|\mathbf{u}\|^4_{L^{12}(B_{1+2^{-1}})}\int_\Omega|\nabla\mathbf{u}|^2\eta^2,\\
&\int_\Omega |\mathbf{u}|^2|\Delta Q|^2\eta^2\\
&\quad\le \frac18\int_\Omega|\Delta\nabla Q|^2\eta^2 + C\int_{B_{1+2^{-1}}}|\Delta Q|^2\\
&\qquad + C\|\mathbf{u}\|^4_{L^{12}(B_{1+2^{-1}})}\int_\Omega|\Delta Q|^2\eta^2,\\
&\int_\Omega |\nabla Q|^2|\Delta Q|^2\eta^2\\
&\quad\le \frac18\int_\Omega|\Delta\nabla Q|^2\eta^2 + C\int_{B_{1+2^{-1}}}|\Delta Q|^2\\
&\qquad + C\|\nabla Q\|^4_{L^{12}(B_{1+2^{-1}})}\int_\Omega|\Delta Q|^2\eta^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_\Omega|\nabla Q|^2|\nabla\mathbf{u}|^2\eta^2 \le \frac18\int_\Omega|\nabla\mathbf{u}|^2\eta^2 + C\int_{B_{1+2^{-1}}}|\nabla\mathbf{u}|^2\\
+ C\|\nabla Q\|^4_{L^{12}(B_{1+2^{-1}})}\int_\Omega|\nabla\mathbf{u}|^2\eta^2.
\end{aligned}
$$

Substituting these estimates into the above inequality, we arrive at

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|\nabla\mathbf{u}|^2+|\nabla^2 Q|^2)\eta^2 + \int_\Omega(|\nabla^2\mathbf{u}|^2+|\Delta\nabla Q|^2)\eta^2\\
&\quad\le C\int_{B_{1+2^{-1}}}(|\mathbf{u}|^2+|\nabla Q|^2+|\nabla\mathbf{u}|^2+|\Delta Q|^2+|P|^2)\\
&\qquad + C(1+\|(\mathbf{u},\nabla Q)\|^{12}_{L^{12}(B_{1+2^{-1}})})\int_\Omega(|\nabla\mathbf{u}|^2+|\nabla^2 Q|^2)\eta^2. \qquad (5.61)
\end{aligned}
$$

From (5.59), we can apply Gronwall's inequality to (5.61) to show that (5.58) holds for $k = 1$. For $k \ge 2$, we can perform an induction argument as in [18] Lemma 3.4. We leave the details to interested readers.

It is readily seen that by the Sobolev embedding theorem, Lemma 5.3 implies that $(\nabla^k u, \nabla^{k+1} Q) \in L^\infty(\mathbb{P}_{\frac{r_0}{4}}(z_0))$ for any $k \geq 1$. This, combined with the theory of the linear Stokes equation and the heat equation, would imply the smoothness of $(\mathbf{u}, Q)$ in $\mathbb{P}_{\frac{r_0}{4}}(z_0)$. This completes the proof. □

Applying Lemma 5.3, we can prove a weaker version of Theorem 1.1.

**Proposition 5.1.** *Under the same assumptions as in Theorem 1.1, there exists a closed subset* $\Sigma \subset \Omega \times (0, \infty)$, *with* $\mathcal{P}^{\frac{5}{3}}(\Sigma) = 0$, *such that* $(\mathbf{u}, Q) \in C^\infty(\Omega \times (0, \infty) \setminus \Sigma)$.

*Proof.* First it follows from Lemma 4.1 and Lemma 3.2 that for any $\delta > 0$, $Q$ and $f_{\mathrm{BM}}(Q)$ are bounded in $\Omega \times (\delta, \infty)$. Define

$$\Sigma_\delta = \Big\{ z \in \Omega \times (\delta, \infty) : \liminf_{r \to 0} r^{-2} \int_{\mathbb{P}_r(z)} (|\mathbf{u}|^3 + |\nabla Q|^3) \,\mathrm{d}x\mathrm{d}t + \big(r^{-2} \int_{\mathbb{P}_r(z)} |P|^{\frac{3}{2}} \,\mathrm{d}x\mathrm{d}t\big)^2 > \varepsilon_0^3 \Big\}.$$

From Lemma 5.3, we know that $\Sigma_\delta$ is closed and $(\mathbf{u}, Q) \in C^\infty(\Omega \times (\delta, \infty) \setminus \Sigma_\delta)$. Since $\delta > 0$ is arbitrary, we have that $(\mathbf{u}, Q) \in C^\infty(\Omega \times (0, \infty) \setminus \cup_{\delta > 0} \Sigma_\delta)$.

Since $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(\Omega \times (0, \infty))$ and $\nabla Q \in L_t^\infty H_x^1 \cap L_t^2 H_x^2(\Omega \times (0, \infty))$, we see that $(\mathbf{u}, \nabla Q) \in L^{\frac{10}{3}}(\Omega \times (0, \infty))$. Moreover, since $P$ solves the Poisson equation (5.57) in $\Omega \times (0, \infty)$, we conclude that $P \in L^{\frac{5}{3}}(\Omega \times (0, \infty))$. By Hölder's inequality, we see that $\Sigma_\delta$ is a subset of

$$\mathcal{S}_\delta = \Big\{ z \in \Omega \times (\delta, \infty) : \liminf_{r \to 0} r^{-\frac{5}{3}} \int_{\mathbb{P}_r(z)} (|\mathbf{u}|^{\frac{10}{3}} + |\nabla Q|^{\frac{10}{3}}) \,\mathrm{d}x\mathrm{d}t + \big(r^{-\frac{5}{3}} \int_{\mathbb{P}_r(z)} |P|^{\frac{5}{3}} \,\mathrm{d}x\mathrm{d}t\big)^2 > \varepsilon_0^{\frac{10}{3}} \Big\}.$$

A simple covering argument implies that $\mathcal{P}^{\frac{5}{3}}(\mathcal{S}_\delta) = 0$, see [32]. Hence $\Sigma = \cup_{\delta > 0} \Sigma_\delta$ has $\mathcal{P}^{\frac{5}{3}}(\Sigma) = 0$. This completes the proof. □

## 6. Partial Regularity, part II

In this section, we will utilize the results from the previous section and the Sobolev inequality to first show the so-called A-B-C-D Lemmas (see [5] and [23]) and then establish an improved $\varepsilon_1$-regularity property for suitable weak solutions to (1.6).

**Theorem 6.1.** *Under the same assumptions as in Theorem 1.1, there exists* $\varepsilon_1 > 0$ *such that if* $(\mathbf{u}, Q) : \Omega \times (0, \infty) \mapsto \mathbb{R}^3 \times \mathcal{S}_0^{(3)}$ *is a suitable weak solution of* (1.5), *which satisfies, for* $z_0 \in \Omega \times (0, \infty)$,

$$\limsup_{r \to 0} \frac{1}{r} \int_{\mathbb{P}_r(z_0)} \big(|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2\big) \mathrm{d}x\mathrm{d}t < \varepsilon_1^2, \tag{6.1}$$

*then* $(\mathbf{u}, Q)$ *is smooth near* $z_0$.

For simplicity, we assume that $z_0 = (0,0) \in \Omega \times (0,\infty)$. To streamline the presentation, we introduce the following dimensionless quantities:

$$\begin{aligned}
A(r) &:= \sup_{-r^2 \leqq t \leqq 0} r^{-1} \int_{B_r(0)\times\{t\}} (|\mathbf{u}|^2 + |\nabla Q|^2)\, \mathrm{d}x, \\
B(r) &:= \frac{1}{r} \int_{\mathbb{P}_r(0,0)} (|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2)\, \mathrm{d}x\mathrm{d}t, \\
C(r) &:= \frac{1}{r^2} \int_{\mathbb{P}_r(0,0)} (|\mathbf{u}|^3 + |\nabla Q|^3)\, \mathrm{d}x\mathrm{d}t, \\
D(r) &:= r^{-2} \int_{\mathbb{P}_r(0,0)} |P|^{\frac{3}{2}}\, \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

We also set

$$(\mathbf{u})_r(t) := \frac{1}{|B_r(0)|} \int_{B_r(0)} \mathbf{u}(x,t)\, \mathrm{d}x,\ (\nabla Q)_r(t) := \frac{1}{|B_r(0)|} \int_{B_r(0)} \nabla Q(x,t)\, \mathrm{d}x.$$

We also let $A \lesssim B$ to denote $A \leq cB$ for some universal positive constant $c > 0$.

We recall the following interpolation Lemma, whose proof can be found in [5]:

**Lemma 6.1.** *For $v \in H^1(\mathbb{R}^3)$,*

$$\begin{aligned}
\int_{B_r(0)} |v|^q(x,t)\, \mathrm{d}x \lesssim \Big(\int_{B_r(0)} |\nabla v|^2(x,t)\, \mathrm{d}x\Big)^{\frac{q}{2}-a} \Big(\int_{B_r(0)} |v|^2(x,t)\, \mathrm{d}x\Big)^a \\
+ r^{3\left(1-\frac{q}{2}\right)} \Big(\int_{B_r(0)} |v|^2(x,t)\, \mathrm{d}x\Big)^{\frac{q}{2}}.
\end{aligned} \tag{6.2}$$

*for every $B_r(0) \subset \mathbb{R}^3$, $2 \leqq q \leqq 6$, $a = \frac{3}{2}\left(1 - \frac{q}{6}\right)$.*

Applying Lemma 6.1, we can have

**Lemma 6.2.** *For any $\mathbf{u} \in L^\infty([-\rho^2, 0], L^2(B_\rho(0))) \cap L^2([-\rho^2, 0], H^1(B_\rho(0)))$, and $Q \in L^\infty([-\rho^2, 0], H^1(B_\rho(0))) \cap L^2([-\rho^2, 0], H^2(B_\rho(0)))$, it holds that for any $0 < r \leqq \rho$,*

$$C(r) \lesssim \Big(\frac{r}{\rho}\Big)^3 A^{\frac{3}{2}}(\rho) + \Big(\frac{\rho}{r}\Big)^3 A^{\frac{3}{4}}(\rho) B^{\frac{3}{4}}(\rho). \tag{6.3}$$

*Proof.* From (6.1) with $q = 3$, $a = \frac{3}{4}$, we obtain that, for any $v \in H^1(B_\rho(0))$,

$$\begin{aligned}
\int_{B_r(0)} |v|^3(x,t)\, \mathrm{d}x \lesssim \Big(\int_{B_r(0)} |\nabla v|^2(x,t)\, dx\Big)^{\frac{3}{4}} \Big(\int_{B_r(0)} |v|^2(x,t)\, \mathrm{d}x\Big)^{\frac{3}{4}} \\
+ r^{-\frac{3}{2}} \Big(\int_{B_r(0)} |v|^2(x,t)\, \mathrm{d}x\Big)^{3/2}.
\end{aligned} \tag{6.4}$$

Applying Poincaré's inequality, we obtain that for $0 < r \le \rho$,

$$
\begin{aligned}
&\int_{B_r(0)} (|\mathbf{u}|^2 + |\nabla Q|^2)\,\mathrm{d}x \\
&\quad \lesssim \int_{B_r(0)} \Big(\Big||\mathbf{u}|^2 - (|\mathbf{u}|^2)_\rho\Big| + \Big||\nabla Q|^2 - (|\nabla Q|^2)_\rho\Big|\Big)\,\mathrm{d}x + \big(\frac{r}{\rho}\big)^3 \int_{B_\rho(0)} (|\mathbf{u}|^2 + |\nabla Q|^2)\,\mathrm{d}x \\
&\quad \lesssim \rho \int_{B_\rho(0)} (|\mathbf{u}||\nabla \mathbf{u}| + |\nabla Q||\nabla^2 Q|)\,\mathrm{d}x + \big(\frac{r}{\rho}\big)^3 \int_{B_\rho(0)} (|\mathbf{u}|^2 + |\nabla Q|^2)\,\mathrm{d}x \\
&\quad \lesssim \rho^{\frac{3}{2}} \big(\rho^{-1} \int_{B_\rho(0)} (|\mathbf{u}|^2 + |\nabla Q|^2)\,\mathrm{d}x\big)^{\frac{1}{2}} \big(\int_{B_\rho(0)} (|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2)\,\mathrm{d}x\big)^{\frac{1}{2}} \\
&\qquad + \big(\frac{r}{\rho}\big)^3 \int_{B_\rho(0)} (|\mathbf{u}|^2 + |\nabla Q|^2)\,\mathrm{d}x \\
&\quad \lesssim \rho^{\frac{3}{2}} A^{\frac{1}{2}}(\rho) \big(\int_{B_\rho(0)} (|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2)\,\mathrm{d}x\big)^{\frac{1}{2}} + \big(\frac{r}{\rho}\big)^3 \rho A(\rho).
\end{aligned}
$$

Substituting this estimate into the second term of the right hand side of the previous inequality, we conclude that

$$
\begin{aligned}
&\int_{B_r(0)} \Big(|\mathbf{u}|^3 + |\nabla Q|^3\Big)\,\mathrm{d}x \\
&\quad \lesssim \rho^{\frac{3}{4}} \big(\int_{B_r(0)} \Big(|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2\Big)\,\mathrm{d}x\big)^{\frac{3}{4}} \big(\rho^{-1} \int_{B_r(0)} (|\mathbf{u}|^2 + |\nabla Q|^2)(x,t)\,\mathrm{d}x\big)^{\frac{3}{4}} \\
&\qquad + r^{-\frac{3}{2}} \big(\int_{B_r(0)} (|\mathbf{u}|^2 + |\nabla Q|^2)(x,t)\,\mathrm{d}x\big)^{\frac{3}{2}} \\
&\quad \lesssim \rho^{\frac{3}{4}} A^{\frac{3}{4}}(\rho) \big(\int_{B_r(0)} (|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2)(x,t)\,\mathrm{d}x\big)^{\frac{3}{4}} \\
&\qquad + r^{-\frac{3}{2}} \big(\int_{B_r(0)} (|\mathbf{u}|^2 + |\nabla Q|^2)(x,t)\,\mathrm{d}x\big)^{\frac{3}{2}} \\
&\quad \lesssim \big(\rho^{\frac{3}{4}} + \frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}}\big) \big(\int_{B_r(0)} (|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2)\,\mathrm{d}x\big)^{\frac{3}{4}} A^{\frac{3}{4}}(\rho) + \big(\frac{r}{\rho}\big)^3 A^{\frac{3}{2}}(\rho).
\end{aligned}
$$

Integrating this inequality over $[-r^2, 0]$, by Hölder's inequality, we have

$$
\begin{aligned}
C(r) &= \frac{1}{r^2} \int_{\mathbb{P}_r(0,0)} (|\mathbf{u}|^3 + |\nabla Q|^3)\,\mathrm{d}x \\
&\lesssim \big(\frac{r}{\rho}\big)^3 A^{\frac{3}{2}}(\rho) + \big(\rho^{\frac{3}{4}} + \frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}}\big) \int_{-r^2}^{0} \big(\int_{B_r(0)} (|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2)\,\mathrm{d}x\big)^{\frac{3}{4}}\,\mathrm{d}t\, A^{\frac{3}{4}}(\rho) \\
&\lesssim \big(\frac{r}{\rho}\big)^3 A^{\frac{3}{2}}(\rho) + r^{-\frac{3}{2}} \rho^{\frac{3}{4}} \big(\rho^{\frac{3}{4}} + \frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}}\big) A^{\frac{3}{4}}(\rho) B^{\frac{3}{4}}(\rho) \\
&\lesssim \big(\frac{r}{\rho}\big)^3 A^{\frac{3}{2}}(\rho) + \big[\big(\frac{\rho}{r}\big)^{\frac{3}{2}} + \big(\frac{\rho}{r}\big)^3\big] A^{\frac{3}{4}}(\rho) B^{\frac{3}{4}}(\rho) \\
&\lesssim \big(\frac{r}{\rho}\big)^3 A^{\frac{3}{2}}(\rho) + \big(\frac{\rho}{r}\big)^3 A^{\frac{3}{4}}(\rho) B^{\frac{3}{4}}(\rho).
\end{aligned}
$$

This completes the proof of (5.2). □

Next we want to estimate the pressure function.

**Lemma 6.3.** *Under the same assumption as for Lemma 6.2, it holds that for any* $0 < r \leqq \frac{\rho}{2}$,

$$D(r) \lesssim \frac{r}{\rho} D(\rho) + \left(\frac{\rho}{r}\right)^2 A^{\frac{3}{4}}(\rho) B^{\frac{3}{4}}(\rho). \tag{6.5}$$

*Proof.* From the scaling invariance of all quantities, we only need to consider the case $\rho = 1$, $0 < r \leqq \frac{1}{2}$. By taking divergence of the equation $(1.5)_1$, we obtain

$$\begin{aligned}
-\Delta P &= \mathrm{div}^2 \left[\mathbf{u} \otimes \mathbf{u} + \nabla Q \otimes \nabla Q\right] \\
&= \mathrm{div}^2 \left[(\mathbf{u} - (\mathbf{u})_1) \otimes (\mathbf{u} - (\mathbf{u})_1) + \nabla Q \otimes \nabla Q\right] \\
&= \mathrm{div}^2 [(\mathbf{u} - (\mathbf{u})_1) \otimes (\mathbf{u} - (\mathbf{u})_1) + (\nabla Q - (\nabla Q)_1) \otimes (\nabla Q - (\nabla Q)_1)] \\
&\quad + \mathrm{div}^2 [(\nabla Q)_1 \otimes (\nabla Q - (\nabla Q)_1) + (\nabla Q - (\nabla Q)_1) \otimes (\nabla Q)_1].
\end{aligned} \tag{6.6}$$

Let $\eta \in C_0^\infty(\mathbb{R}^3)$ be a cut off function of $B_{\frac{1}{2}}(0)$ such that

$$\begin{cases}
\eta = 1, & \text{in } B_{\frac{1}{2}}(0), \\
\eta = 0, & \text{in } \mathbb{R}^3 \setminus B_{\frac{3}{4}}(0), \\
0 \leqq \eta \leqq 1, \ |\nabla \eta| \leqq 8.
\end{cases} \tag{6.7}$$

Define the following auxillary function:

$$\begin{aligned}
P_1(x, t) = -\int_{\mathbb{R}^3} & \nabla_y^2 G(x - y) : \eta^2(y) \big[(\mathbf{u} - (\mathbf{u})_1) \otimes (\mathbf{u} - (\mathbf{u})_1) \\
& + (\nabla Q - (\nabla Q)_1) \otimes (\nabla Q - (\nabla Q)_1) + (\nabla Q - (\nabla Q)_1) \otimes (\nabla Q)_1 \\
& + (\nabla Q)_1 \otimes (\nabla Q - (\nabla Q)_1)\big](y, t)\, \mathrm{d}y.
\end{aligned}$$

Then we have

$$-\Delta P_1 = \mathrm{div}^2 \left[(\mathbf{u} - (\mathbf{u})_1) \otimes (\mathbf{u} - (\mathbf{u})_1) + \nabla Q \otimes \nabla Q\right] \ \text{in } B_{\frac{1}{2}}(0),$$

and

$$-\Delta (P - P_1) = 0 \text{ in } B_{\frac{1}{2}}(0).$$

For $P_1$, we apply the Calderon-Zygmund theory to deduce that

$$\begin{aligned}
\|P_1\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^{\frac{3}{2}} &\lesssim \left\|\eta^2 |\mathbf{u} - (\mathbf{u})_1|^2\right\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^{\frac{3}{2}} + \left\|\eta^2 |\nabla Q - (\nabla Q)_1|^2\right\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^{\frac{3}{2}} \\
&\quad + \left\|\eta^2 |(\nabla Q)_1| |\nabla Q - (\nabla Q)_1|\right\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^{\frac{3}{2}} \\
&\lesssim \int_{B_1(0)} (|\mathbf{u} - (\mathbf{u})_1|^3 + |\nabla Q - (\nabla Q)_1|^3)\, \mathrm{d}x \\
&\quad + |(\nabla Q)_1|^{\frac{3}{2}} \int_{B_1(0)} |\nabla Q - (\nabla Q)_1|^{\frac{3}{2}}\, \mathrm{d}x.
\end{aligned} \tag{6.8}$$

Since $P - P_1$ is harmonic in $B_{\frac{1}{2}}(0)$, we get
$\frac{1}{r^2}\|P - P_1\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(B_r(0))} \lesssim r\|P - P_1\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(B_1(0))} \lesssim r\big(\|P\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(B_1(0))} + \|P_1\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(B_1(0))}\big).$
Integrating it over $[-r^2, 0]$ and applying (5.8), we can show that

$$\begin{aligned}
&\frac{1}{r^2}\int_{\mathbb{P}_r(0,0)} |P|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t\\
&\lesssim r\int_{\mathbb{P}_1(0,0)} |P|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t + \frac{1}{r^2}\int_{\mathbb{P}_1(0,0)} (|\mathbf{u} - (\mathbf{u})_1|^3 + |\nabla Q - (\nabla Q)_1|^3)\,\mathrm{d}x\mathrm{d}t\\
&\quad + \frac{1}{r^2}\Big(\sup_{-1\le t\le 0} |(\nabla Q)_1(t)|\Big)^{\frac{3}{2}}\int_{\mathbb{P}_1(0,0)} |\nabla Q - (\nabla Q)_1|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t\\
&\lesssim r\int_{\mathbb{P}_1(0,0)} |P|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t + \frac{1}{r^2}\int_{\mathbb{P}_1(0,0)} (|\mathbf{u} - (\mathbf{u})_1|^3 + |\nabla Q - (\nabla Q)_1|^3)\,\mathrm{d}x\mathrm{d}t\\
&\quad + \frac{1}{r^2}A^{\frac{3}{4}}(1)\int_{\mathbb{P}_1(0,0)} |\nabla Q - (\nabla Q)_1|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

This, combined with the interpolation inequality

$$\begin{aligned}
&\int_{\mathbb{P}_1(0,0)} (|\mathbf{u} - (\mathbf{u})_1|^3 + |\nabla Q - (\nabla Q)_1|^3)\,\mathrm{d}x\mathrm{d}t\\
&\quad\lesssim \sup_{-1\leqq t\leqq 0}\Big(\int_{B_1(0)} (|\mathbf{u}|^2 + |\nabla Q|^2)\,\mathrm{d}x\Big)^{\frac{3}{4}} \times \Big(\int_{\mathbb{P}_1(0,0)} (|\nabla\mathbf{u}|^2 + |\nabla^2 Q|^2)\,\mathrm{d}x\mathrm{d}t\Big)^{\frac{3}{4}},
\end{aligned}$$

and Hölder's inequality

$$\int_{\mathbb{P}_1(0,0)} |\nabla Q - (\nabla Q)_1|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t \lesssim \Big(\int_{\mathbb{P}_1(0,0)} |\nabla Q - (\nabla Q)_1|^2\,\mathrm{d}x\mathrm{d}t\Big)^{\frac{3}{4}},$$

implies that

$$D(r) \lesssim rD(1) + \frac{1}{r^2}A^{\frac{3}{4}}(1)B^{\frac{3}{4}}(1).$$

This, after scaling back to $\rho$, yields (6.5). The proof is now complete. □

*Proof of Theorem 6.1*. For $\theta \in (0, \frac{1}{2})$ and $\rho \in (0, 1)$, let $\varphi \in C_0^\infty(\mathbb{P}_{\theta\rho}(0, 0))$ be a function such that

$$\varphi = 1 \text{ in } \mathbb{P}_{\frac{\theta\rho}{2}}(0, 0),\ |\nabla\varphi| \lesssim \frac{1}{\theta\rho},\ |\nabla^2\varphi| + |\varphi_t| \lesssim (\frac{1}{\theta\rho})^2.$$

Applying the local energy inequality in Lemma 2.2, the maximum principles Lemmas 4.1 and 3.2 , and the integration by parts, we obtain that

$$\begin{aligned}
&\sup_{-(\theta\rho)^2\leqq t\leqq 0}\int_\Omega (|\mathbf{u}|^2 + |\nabla Q|^2)\varphi^2\,\mathrm{d}x + \int_{\Omega\times[-(\theta\rho)^2,0]} (|\nabla\mathbf{u}|^2 + |\nabla^2 Q|^2)\varphi^2\,\mathrm{d}x\mathrm{d}t\\
&\lesssim \int_{\Omega\times[-(\theta\rho)^2,0]} (|\mathbf{u}|^2 + |\nabla Q|^2)(|\varphi_t| + |\nabla\varphi|^2 + |\nabla^2\varphi|)\,\mathrm{d}x\mathrm{d}t\\
&\quad + \int_{\Omega\times[-(\theta\rho)^2,0]} [(|\mathbf{u}|^2 - (|\mathbf{u}|^2)_{\theta\rho}) + (|\nabla Q|^2 - |\nabla Q|^2)_{\theta\rho}) + |P|]|\mathbf{u}||\nabla\varphi|\,\mathrm{d}x\mathrm{d}t\\
&\quad + \int_{\Omega\times[-(\theta\rho)^2,0]} |\nabla Q|^2\varphi^2\,\mathrm{d}x\mathrm{d}t + \int_{\Omega\times[-(\theta\rho)^2,0]} (|\nabla\mathbf{u}||\nabla Q| + |\mathbf{u}||\Delta Q|)|\varphi||\nabla\varphi|\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

This, with the help of Young's inequality, gives

$$\begin{aligned}
&\int_{\Omega\times[-(\theta\rho)^2,0]} (|\nabla \mathbf{u}||\nabla Q| + |\mathbf{u}||\Delta Q|)|\varphi||\nabla\varphi|\,\mathrm{d}x\mathrm{d}t \\
&\leqq \frac{1}{2}\int_{\Omega\times[-(\theta\rho)^2,0]} (|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2)\varphi^2\,\mathrm{d}x\mathrm{d}t \\
&\quad + 4\int_{\Omega\times[-(\theta\rho)^2,0]} (|\mathbf{u}|^2 + |\nabla Q|^2)|\nabla\varphi|^2\,\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

which implies that

$$\begin{aligned}
&A(\frac{1}{2}\theta\rho) + B(\frac{1}{2}\theta\rho) \\
&= \sup_{-(\frac{\theta\rho}{2})^2\leqq t\leqq 0} \frac{2}{\theta\rho}\int_{B_{\frac{\theta\rho}{2}}(0)} (|\mathbf{u}|^2 + |\nabla Q|^2)\,\mathrm{d}x + \frac{2}{\theta\rho}\int_{\mathbb{P}_{\frac{\theta\rho}{2}}(0,0)} (|\nabla\mathbf{u}|^2 + |\nabla^2 Q|^2)\,\mathrm{d}x\mathrm{d}t \\
&\lesssim \sup_{-(\theta\rho)^2\leqq t\leqq 0} \frac{1}{\theta\rho}\int_{\mathbb{R}^3} (|\mathbf{u}|^2 + |\nabla Q|^2)\varphi^2\,\mathrm{d}x + \frac{1}{\theta\rho}\int_{\mathbb{R}^3\times[-(\theta\rho)^2,0]} (|\nabla\mathbf{u}|^2 + |\nabla^2 Q|^2)\varphi^2\,\mathrm{d}x\mathrm{d}t \\
&\lesssim \frac{1}{\theta\rho}\int_{\mathbb{R}^3\times[-(\theta\rho)^2,0]} (|\mathbf{u}|^2 + |\nabla Q|^2)(|\varphi_t| + |\nabla\varphi|^2 + |\nabla^2\varphi|)\,\mathrm{d}x\mathrm{d}t \\
&\quad + \frac{1}{\theta\rho}\int_{\mathbb{R}^3\times[-(\theta\rho)^2,0]} [(|\mathbf{u}|^2 - (|\mathbf{u}|^2)_{\theta\rho}) + (|\nabla Q|^2 - (|\nabla Q|^2)_{\theta\rho}) + |P|]|\mathbf{u}||\nabla\varphi|\,\mathrm{d}x\mathrm{d}t \\
&\quad + \frac{1}{\theta\rho}\int_{\mathbb{R}^3\times[-(\theta\rho)^2,0]} |\nabla Q|^2\varphi^2\,\mathrm{d}x\mathrm{d}t \\
&q \lesssim \frac{1}{(\theta\rho)^3}\int_{\mathbb{P}_{\theta\rho}(0,0)} (|\mathbf{u}|^2 + |\nabla Q|^2)\,\mathrm{d}x\mathrm{d}t + \frac{1}{(\theta\rho)^2}\int_{\mathbb{P}_{\theta\rho}(0,0)} |P||\mathbf{u}|\,\mathrm{d}x\mathrm{d}t \\
&\quad + \frac{1}{(\theta\rho)^2}\int_{\mathbb{P}_{\theta\rho}(0,0)} \big(||\mathbf{u}|^2 - (|\mathbf{u}|^2)_{\theta\rho}| + ||\nabla Q|^2 - (|\nabla Q|^2)_{\theta\rho}|\big)\,|\mathbf{u}|\,\mathrm{d}x\mathrm{d}t \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

It is not hard to see that

$$\begin{aligned}
|I_1| &\lesssim \big(\frac{1}{(\theta\rho)^2}\int_{\mathbb{P}_{\theta\rho}(0,0)} (|\mathbf{u}|^3 + |\nabla Q|^3)\,\mathrm{d}x\mathrm{d}t\big)^{\frac{2}{3}} \lesssim C^{\frac{2}{3}}(\theta\rho), \\
|I_2| &\lesssim \big(\frac{1}{(\theta\rho)^2}\int_{\mathbb{P}_{\theta\rho}(0,0)} |\mathbf{u}|^3\,\mathrm{d}x\mathrm{d}t\big)^{\frac{1}{3}}\big(\frac{1}{(\theta\rho)^2}\int_{\mathbb{P}_{\theta\rho}(0,0)} |P|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t\big)^{\frac{2}{3}} \lesssim C^{\frac{1}{3}}(\theta\rho)D^{\frac{2}{3}}(\theta\rho),
\end{aligned}$$

while, by employing Hölder's and Poincaré's inequalities, we have

$$\begin{aligned}
|I_3| &\lesssim \frac{1}{(\theta\rho)^2}\int_{-(\theta\rho)^2}^0\int_{B_{\theta\rho}(0)} (|\mathbf{u}||\nabla\mathbf{u}| + |\nabla Q||\nabla^2 Q|)\big(\int_{B_{\theta\rho}(0)} |\mathbf{u}|^3 + |\nabla Q|^3\big)^{\frac{1}{3}}\,\mathrm{d}t \\
&\lesssim A^{\frac{1}{2}}(\theta\rho)B^{\frac{1}{2}}(\theta\rho)C^{\frac{1}{3}}(\theta\rho).
\end{aligned}$$

Putting together all the estimates, we have

$$\begin{aligned}
A(\frac{1}{2}\theta\rho) + B(\frac{1}{2}\theta\rho) &\lesssim \big[C^{\frac{2}{3}}(\theta\rho) + A^{\frac{1}{2}}(\theta\rho)B^{\frac{1}{2}}(\theta\rho)C^{\frac{1}{3}}(\theta\rho) + C^{\frac{1}{3}}(\theta\rho)D^{\frac{2}{3}}(\theta\rho)\big] \\
&\lesssim \big[C^{\frac{2}{3}}(\theta\rho) + A(\theta\rho)B(\theta\rho) + D^{\frac{4}{3}}(\theta\rho)\big],
\end{aligned}$$

so that

$$A^{\frac{3}{2}}(\frac{1}{2}\theta\rho) \lesssim \big[C(\theta\rho) + A^{\frac{3}{2}}(\theta\rho)B^{\frac{3}{2}}(\theta\rho) + D^2(\theta\rho)\big],$$

while

$$D^2(\theta\rho) \lesssim \theta^2\big[D^2(\rho) + \theta^{-6}A^{\frac{3}{2}}(\rho)B^{\frac{3}{2}}(\rho)\big],$$

and

$$C(\theta\rho) \lesssim \theta^3 A^{\frac{3}{2}}(\rho) + \theta^{-3}A^{\frac{3}{4}}(\rho)B^{\frac{3}{4}}(\rho).$$

Also note that

$$A^{\frac{3}{2}}(\theta\rho)B^{\frac{3}{2}}(\theta\rho) \le \theta^{-3}A^{\frac{3}{2}}(\rho)B^{\frac{3}{2}}(\rho).$$

Therefore we conclude that for $0 < \theta_0 < \frac{1}{2}$,

$$\begin{aligned}
&A^{\frac{3}{2}}(\frac{1}{2}\theta_0\rho) + D^2(\frac{1}{2}\theta_0\rho)\\
&\quad \leqq c[\theta_0^2 D^2(\rho) + (\theta_0^{-3} + \theta_0^{-4})A^{\frac{3}{2}}(\rho)B^{\frac{3}{2}}(\rho) + \theta_0^3 A^{\frac{3}{2}}(\rho) + \theta_0^{-3}A^{\frac{3}{4}}(\rho)B^{\frac{3}{4}}(\rho)]\\
&\quad \leqq c[\theta_0^2(D^2(\rho) + A^{\frac{3}{2}}(\rho)) + \theta_0^{-8}A^{\frac{3}{2}}(\rho)B^{\frac{3}{2}}(\rho) + \theta_0^2]\\
&\quad \leqq c(\theta_0^2 + \theta_0^{-8}B^{\frac{3}{2}}(\rho))(A^{\frac{3}{2}}(\rho) + D^2(\rho)) + c\theta_0^2.
\end{aligned}$$

For $\varepsilon_1 > 0$ given by Theorem 5.1, let $\theta_0 \in (0, \frac{1}{2})$ such that

$$c\theta_0^2 = \min\big\{\frac{1}{4}, \frac{1}{2}\varepsilon_1^2\big\}.$$

From (6.1), we know that

$$\limsup_{\rho\to 0} B(\rho) \le \varepsilon_1^2,$$

hence there exists $\rho_0 > 0$ such that

$$c\theta_0^{-8}B^{\frac{3}{2}}(\rho) \le \frac{1}{4},\ \forall 0 < \rho < \rho_0.$$

Therefore we conclude that there exist $\theta_0 \in (0, \frac{1}{2})$ and $\rho_0 > 0$ such that

$$A^{\frac{3}{2}}(\frac{1}{2}\theta_0\rho) + D^2(\frac{1}{2}\theta_0\rho) \le \frac{1}{2}(A^{\frac{3}{2}}(\rho) + D^2(\rho)) + \frac{1}{2}\varepsilon_1^2,\ \forall 0 < \rho < \rho_0.$$

Iterating this inequality yields that

$$A^{\frac{3}{2}}((\frac{1}{2}\theta_0)^k\rho) + D^2((\frac{1}{2}\theta_0)^k\rho) \le \frac{1}{2^k}(A^{\frac{3}{2}}(\rho) + D^2(\rho)) + \varepsilon_1^2 \tag{6.9}$$

holds for all $0 < \rho < \rho_0$ and $k \ge 1$.

Employing (5.2) and (6.9), we obtain that

$$C((\frac{1}{2}\theta_0)^k\rho) \le c\Big[(\frac{1}{2}\theta_0)^3 A^{\frac{3}{2}}((\frac{1}{2}\theta_0)^{k-1}\rho) + (\frac{1}{2}\theta_0)^{-3} A^{\frac{3}{4}}((\frac{1}{2}\theta_0)^{k-1}\rho) B^{\frac{3}{4}}((\frac{1}{2}\theta_0)^{k-1}\rho)\Big]$$
$$\le c\Big[(\frac{1}{2}\theta_0)^3 + (\frac{1}{2}\theta_0)^{-3}\varepsilon_1^{\frac{3}{2}}\Big]\Big[\frac{1}{2^{k-1}}(A^{\frac{3}{2}}(\rho) + D^2(\rho)) + \varepsilon_1^2\Big] \tag{6.10}$$

holds for all $0 < \rho < \rho_0$ and $k \ge 1$.

Putting (6.9) and (6.10) together, we obtain that

$$\limsup_{k\to\infty}\Big[C((\frac{1}{2}\theta_0)^k\rho) + D^2((\frac{1}{2}\theta_0)^k\rho)\Big] \le c\Big[1 + (\frac{1}{2}\theta_0)^3 + (\frac{1}{2}\theta_0)^{-3}\varepsilon_1^{\frac{3}{2}}\Big]\varepsilon_1^2 \le \frac{1}{2}\varepsilon_0^3 \tag{6.11}$$

holds for all $\rho \in (0, \rho_0)$, provided $\varepsilon_1 = \varepsilon_1(\theta_0, \varepsilon_0) > 0$ is chosen sufficiently small. Therefore, by Lemma 5.4 $(\mathbf{u}, Q, P)$ is smooth near $(0, 0)$. This completes the proof. □

Theorem 1.1 can be proved by the following covering argument. Let $\Sigma$ be the singular set of suitable weak solutions $(\mathbf{u}, Q, P)$. If $(x, t) \in \Sigma$, then, by theorem 6.1,

$$\limsup_{r\to 0}\frac{1}{r}\int_{\mathbb{P}_r(x,t)}(|\nabla\mathbf{u}|^2 + |\nabla^2 Q|^2)\,\mathrm{d}x\mathrm{d}t \geqq \varepsilon_1. \tag{6.12}$$

Let $V$ be a neighborhood of $\Sigma$ and $\delta > 0$ such that for all $(x, t) \in \Sigma$, we can find $r < \delta$ such that $\mathbb{P}_r(x, t) \subset V$ and

$$\frac{1}{r}\int_{\mathbb{P}_r(x,t)}\left(|\nabla\mathbf{u}|^2 + |\nabla^2 Q|^2\right)\mathrm{d}x\mathrm{d}t \geqq \varepsilon_1.$$

By Vitali's covering lemma, $\exists (x_i, t_i) \in V, 0 < r_i < \delta$ such that $\left\{\mathbb{P}_{r_i}(x_i, t_i)\right\}_{i=1}^{\infty}$ are pairwise disjoint and

$$\Sigma \subset \bigcup_{i=1}^{\infty}\mathbb{P}_{5r_i}(x_i, t_i).$$

Hence

$$\mathcal{P}^1_{5\delta}(\Sigma) \le \sum_{i=1}^{\infty} 5r_i \leqq \frac{5}{\varepsilon_1}\sum_{i=1}^{\infty}\int_{\mathbb{P}_{r_i}(x_i,t_i)}\left(|\nabla\mathbf{u}|^2 + |\nabla^2 Q|^2\right)\mathrm{d}x\mathrm{d}t$$
$$\leqq \frac{5}{\varepsilon_1}\int_{\cup_i\mathbb{P}_{r_i}(x_i,t_i)}\left(|\nabla\mathbf{u}|^2 + |\nabla^2 Q|^2\right)\mathrm{d}x\mathrm{d}t$$
$$\leqq \frac{5}{\varepsilon_1}\int_V\left(|\nabla\mathbf{u}|^2 + |\nabla^2 Q|^2\right)\mathrm{d}x\mathrm{d}t < \infty.$$

We can conclude that $\Sigma$ is of zero Lesbegue measure. Then we can choose $|V|$ to be arbitrarily small from the fact that by

$$\int_0^{\infty}\int_{\Omega}\left(|\nabla\mathbf{u}|^2 + |\nabla^2 Q|^2\right)\mathrm{d}x\mathrm{d}t = \int_0^{\infty}\int_{\Omega}\left(|\nabla\mathbf{u}|^2 + |\Delta Q|^2\right)\mathrm{d}x\mathrm{d}t < \infty$$

and the absolute continuity of the integral, we have

$$\lim_{|V|\to 0}\int_V \left(|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2\right) \mathrm{d}x\mathrm{d}t \to 0.$$

Hence

$$\mathcal{P}^1(\Sigma) = \lim_{\delta\to 0} \mathcal{P}^1_{5\delta}(\Sigma) = 0.$$

This completes the proof of Theorem 1.1. □

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